LEAST SQUARES SMOOTHING AND PREDICTION
UTILIZING TIME DOMAIN FILTERS

DAVID WISNER FISCHER

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DAVID WISNER FISCHER

LEAST SQUARES SMOOTHING AND PREDICTION UTILIZING TIME DOMAIN FILTERS

by

David Wisner Fischer

Lieutenant, United States Navy

Submitted in partial fulfillment
of the requirements
for the degree of
MASTER OF SCIENCE
IN
ENGINEERING ELECTRONICS

United States Naval Postgraduate School
Monterey, California

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PREFACE

Work on this thesis was begun during the first eleven weeks of 1954 while the writer was employed by the electronics design and manufacturing firm of Gilfillan Brothers, Incorporated, located at Los Angeles, California. This employment was the industrial experience tour in the Engineering Electronics Curriculum of the United States Naval Postgraduate School which requires the student to work as a junior engineer in the electronics company of his choice. Completion of the thesis was accomplished during the Spring of 1954 at the United States Naval Postgraduate School.

This investigation was done in cooperation with Mr. Louis A. Ule, a member of the Theoretical Analysis Group in the Engineering Department of Gilfillan Brothers, Incorporated. Its purpose was to round out the least squares smoothing and prediction theory embodied in Mr. Ule's intra-company memorandum of January 12, 1954, entitled, "Polynomial Filters." While a considerable amount of material in the thesis is the result of joint discussion or discovery, the formulation of the weighting function (Chapter II, Section 4) and the proof of the steady state filter response in the absence of noise (Chapter II, Section 5) are abstracts of material written by Mr. Ule.

The writer wishes to express his appreciation to Mr. Ule for his guidance, assistance, and original material. He is also indebted to Professors Abraham Sheingold and Robert Kahal of the United States Naval Postgraduate School Staff for their suggestions and encouragement while acting as thesis advisers and readers.

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TABLE OF SYMBOLS

A	Crosshatched area under the time weighted squared error curve of Figure 4.
a_h	Coefficient of the input function, $f_h(t)$.
b_k	Coefficient of the output function, $g_k(t)$.
$\left[B_{jk}\right]$	The matrix defined by equation (57).
$\left[C_{ki}\right]$	The inverse of the $\left[B_{jk}\right]$ matrix.
$\left[D_{hp}(t)\right]$	The matrix introduced in equation (76).
$e(t)$	The mean-square error.
$f(t)$	An input signal to the filter as a function of time.
$F(s)$	The Laplace transform of $f(t)$.
$g(t)$	An output signal from the filter as a function of time.
$G(s)$	The complex frequency output signal from the filter.
$H(s)$	The complex frequency filter transfer function.
I_j	The integral defined by equation (56).
$\mathcal{L}[\;]$	The Laplace transform of the quantity in the brackets.
m	The order of a polynomial function or the damping constant of an exponential function.
$M(t)$	The squared error weighting function.
$N(t)$	The noise as a function of time.
$P(t)$	The true signal.
$Q(t)$	The "best" approximation of the desired output.
Q_1, Q_2, Q_3	The filter response to the input signals R_1, R_2, and R_3 respectively.
$R(t)$	The noisy input signal to the filter as a function of time.

R_1, R_2, R_3	Input signals defined by equations (137) through (139) inclusive.
t	Time.
T	The smoothing time of a rectangularly weighted filter and the smoothing time constant of an exponentially weighted filter.
$u(t)$	The unit step function.
v	The residual error.
w	The angular frequency of the signal.
$W(t)$	The weighting function of the filter.
x	A dummy variable of integration.
$Z(s)$	A complex frequency input impedance of the filter.
θ	The prediction time.
τ	A dummy variable of integration.

SYNOPSIS

The smoothing and prediction problem is one of extracting from a noisy signal the best estimate of some functional of the true signal at some past present, or future time. The historical development of the least squares solution to the problem is briefly discussed along with other background information relating to this process.

A method of smoothing and prediction is developed which is a dynamic form of least squares curve fitting. The true signal is assumed to be composed of completely foreknown functions of time each multiplied by a constant coefficient. The filter weighting function is chosen so that the filter output will be composed of the same functions as the true signal except for a possible translation in time, differentiation, or integration. The squared error is time weighted before being minimized. While the derivation of the weighting function is independent of the type of noise, the choice of the squared error weighting function may depend upon the character of the noise. Functions which can be treated are found to be the solutions of the reduced equation of a linear differential equation with constant coefficients. The resultant filter weighting function depends upon the true signal, the prediction time, the squared error weighting, and an inverse matrix which is the solution of a set of integral equations.

The resultant filters are constructed of linear elements which do not vary with time. Although no conditions for physical realizability are imposed, a high probability exists that the filters may be constructed by one of several methods. A step by step procedure for the filter design is given together with specific examples of its use. A phenomenon akin to resonance is observed for exponential and polynomial functions.

CHAPTER I INTRODUCTION TO THE SMOOTHING AND PREDICTION PROBLEM

1. Description of the Problem.

The smoothing problem is one of extracting true signals from noisy ones. This problem has been described by Bode and Shannon[2] and by Zadeh and Ragazzini[30]. It is usually assumed that a given perturbed signal, $R(t)$, is the sum of a true signal, $P(t)$, and a random disturbance or noise, $N(t)$.

$$R(t) = P(t) + N(t) \tag{1}$$

By performing appropriate operations on $R(t)$, it is desired to obtain a result $Q(t)$ which is the best approximation of $P(t)$. This smoothing process may be combined with prediction to provide a continuous approximation of $P(t)$ at some past or future time. In other words, given $R(t)$ and a fixed prediction time θ, what is $P(t + \theta)$? θ may be positive, negative, or zero depending upon whether prediction, time delay, or smoothing alone is desired. Both smoothing and prediction may be further combined with differentiation and integration so that in the general case some functional of $P(t + \theta)$ is obtained. Commonly desired functionals are of the form $P(t + \theta)$, $dP(t + \theta)/dt$ and $\int P(t + \theta)\, dt$. The problem of filling the box marked "?" is illustrated in Figure 1.

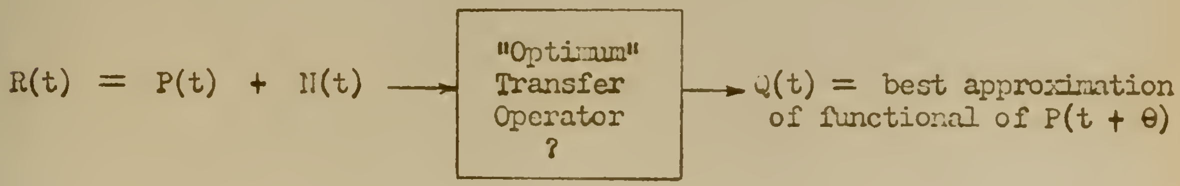

R(t) = P(t) + N(t)
"Optimum" Transfer Operator ?
Q(t) = best approximation of functional of P(t + θ)

Figure 1. The Smoothing and Prediction Problem.

As a class, the physical counterparts of the "optimum" transfer operators are commonly known as smoothing and predicting filters, estimators, or predictors. Since any one filter may not perform all of the aforesaid processes, specific names to indicate exactly what each filter does might be in order. Examples of such names would be "reproducer" to indicate a smoothing process with no time lag or lead; "differentiating predictor" to indicate both differentiation and smoothing with a fixed time lead; and "integrating delayer" to indicate both integration and smoothing with a fixed time delay.

The design of an optimum predictor involves two questions. In the first place, what are the transfer characteristics for the "optimum" transfer operator? Secondly, how may these characteristics be physically realized? Before the first question can be answered, a criterion of performance must be chosen and a method of characterising the signal and noise must be found. By choosing a realistic criterion, the answer to both questions may be greatly simplified. The proper characterization of the signal and noise will permit one to make the optimum use of the a priori information concerning the signal and noise.

While the smoothing and prediction problem is usually thought of as one in communication theory, it might well be generalized to cover such diverse fields as gunnery, economic prediction, weather forecasting and statistics.

2. Performance Criteria.

To attack the problem mathematically, a criterion of performance or measure of fidelity must be chosen for the optimum predictor. This criterion should be realistic in order that the results have meaning.

It should also be simple in order that the mathematical treatment may be economically possible. Normally negative errors are as undesirable as positive ones so that the chosen criterion should be an even function of error.

Graham and Lathrop[8] have examined the criteria for the "optimum" transient response in linear transfer systems called "duplicators," in which the output approximately duplicates the input. They have concluded that the minimum integral of the time-multiplied absolute value of error criterion is best.

$$I = \int_0^\infty t \left| e(t) \right| dt = \text{minimum} \tag{2}$$

The results of this article do not appear to meet the requirements of the smoothing and prediction problem since the lack of a noise component in the input signal has been tacitly assumed.

For signals which consist of a number of discrete pieces of data, the probability that an error exists might be used as a measure of fidelity. It is meaningless in the case of continuous signals in the presence of noise since the probability that an error will occur at any given point is always one.

If the error is designated as e(t) and the mean value of a quantity by a bar above that quantity, a performance criterion for continuous signals might employ the minimization of one of the following:

$$\overline{\left| e(t) \right|}$$

$\overline{e^{2n}(t)}$ where n is a positive integer

$$\overline{\sum_n a_n \, e^{2n}(t)}$$

probability that $|e(t)| >$ a specified limit

probability that mission is not accomplished

The mean-square error criterion introduced by Kharkevitch[18] has found wide usage. This criterion requires that:

$$\overline{e^2(t)} = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} e^2(t)\, dt = \text{minimum} \tag{3}$$

From equation (3) it is evident that the undesirability of an error is weighted according to the square of its magnitude. The principle attention is paid to making the very large errors as small as possible with little regard to small errors. Within the time range of interest the weighting is independent of the time when the error occurs.

There are several reasons why this criterion has found such wide acceptance. First of all, it is generally applicable and in reasonable accord with physical requirements. Numerous cases exist in which the undesirability of an error increases with its magnitude. The mean-square error criterion is reasonable when the error has a normal or Gaussian distribution which is independent of time, a condition ofter encountered in practice. In the second place, the mean-square error method eliminates much of the mathematical complexity involved in the use of other criteria. A highly developed body of mathematical knowledge has been built around the idea of the mean-square value. In some cases, it is the only criterion which is amenable to mathematical treatment. Finally, this criterion leads to useful results. Information can often be obtained from this method even when it is not strictly applicable.

The mean-square criterion does not always meet the requirements of the system. Ragazzini and Zadeh[25] have presented the case in which all

errors larger than a certain limit are equally bad, while those less than the limit are equally acceptable. This is the classic gunnery problem best described by the axiom, "A miss is as good as a mile." Cases may occur in which the undesirability of an error depends upon the time at which the error occurs. For example, the undesirability of an error during a transition from one mode of operation to another may be quite different from that while operating in a given mode. Another case in which the mean-square error criterion would not apply involves the times when as many very accurate predictions as possible are desired and an occasional gross error is of little consequence.

3. Historical Development.

The first successful attempt in this country to put the noise problem on a mathematical basis was Norbert Wiener's[29] classic report for the National Defense Research Council entitled, "The Extrapolation, Interpolation, and Smoothing of Stationary Time Series," which appeared in February 1942. An earlier, similar but not identical article by A. N. Kolmogoroff[19] appeared in a Russian scientific journal in 1941. Although both works are devoted to a study of the optimum prediction problem through the use of statistical methods and support each other, they are the result of independent investigations.

The Wiener-Kolmogoroff theory is based on three assumptions which limit its range of application.

1. The signal and noise may be satisfactorily represented by stationary time series, that is, the statistical properties of the signal and noise do not change with time.

2. The minimum mean-square error is a reasonable criterion of performance. The ensemble average is taken over all possible signal and noise functions, each weighted according to its probability of occurance.

3. The operation used for smoothing and prediction is linear so that it may be performed by a physically realizable filter.

For these reasons the theory has been described as "linear least square prediction and smoothing of stationary time series." If a given time series is stationary and the correlation functions of the signal and noise are known, this theory may be used to obtain the specifications for an optimum predictor.

An early application of Wiener's work by Phillips and Weiss[23] extended the theory to quasi-stationary processes, that is, processes which are stationary in the range of interest. The problem under investigation was the smoothing of data obtained in the tracking of a gunnery target. In this case, the signal and noise characteristics are considered invariant except when the target changes course. Then the error is large enough so that the possibility of a hit is negligible and the results need not be considered.

Because of the formidable mathematics involved in Wiener's theory, Norman Levinson[21, 22] has written two papers with a view toward facilitating the computational procedure and summarizing the significant results. Bode and Shannon[2] have presented a simplified derivation of the theory which does not resort to the use of integral equations or auto-correlation functions. Recent books on information theory, such as those by Goldman[7] and Bell[1], discuss the noise problem.

Zadeh and Ragazzini[30] have further extended the theory in two ways. The desired signal may also contain a non-random function of time which

is representable as a polynomial of degree not greater than n and about which no information other than n is known. In addition, the impulsive response of the predictor may vanish outside a specified time interval $0 \leqq t \leqq T$, whereas T is infinite in Wiener's theory.

A recent paper by Hauser[12] discusses the geometric aspects of the least squares smoothing process.

4. Time versus Frequency Domain Linear Filters.

The most important prerequisite of any filter is that it be physically realizable. Unless this condition is met, any "optimum" transfer operation is of academic interest only. For ease of mathematical treatment and practical realizability, it is usually desirable that the filter also be fixed and linear. James and Weiss[14] have indicated the characteristics of the normal response of a linear filter. The normal response should be a linear function of the input so that the linear superposition of inputs and resultant responses will hold. It should depend only on the past and present values of the input. In addition, this response should be independent of the time origin. This means that the values of circuit elements are fixed and independent of time.

The classical approach to the problem of separating signals from interference, whether due to noise or other signals, is to provide a filter of limited bandwidth as determined by the steady state frequency components of the desired signal. The resonant tuned circuit for sinusoidal signals is a good example of this technique. For signals of a transient nature which have neither bandwidth nor frequency components in the usual sense, this method may be clumsy.

With the increased use of the operational mathematics, it has become

conventional to utilize the Laplace transform when dealing with filters for transient signals. The situation is represented by Figure 2.

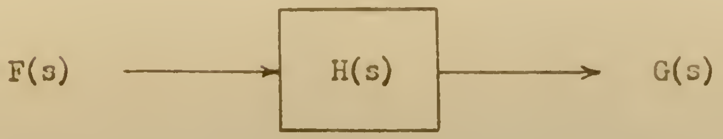


Figure 2. Frequency Domain Filtering.

F(s) and G(s) are the Laplace transforms of the input and output signals respectively. The transfer function of the filter H(s) is defined as the ratio of the Laplace transforms of any normal response and the input that produces it. It follows that

$$G(s) = F(s)\,H(s) \tag{4}$$

The filtering of signals which are functions of time can also be done directly in the time domain. The input signal is f(t) and the filter is completely described by its weighting function, W(t). This weighting function, often called the impulsive response of the filter, is the normal response of the filter to a unit impulse applied at zero time. The weighting function expresses the extent to which the distant past of the input affects the present filter response and hence may be thought of as the memory of the filter. The output of the filter, g(t), is given by the convolution or superposition integral in terms of f(t), W(t), and a dummy variable of integration τ.

$$g(t) = f(t) * W(t) = \int_0^t f(t-\tau)\,W(\tau)\,d\tau \tag{5}$$

Time domain filtering is illustrated in Figure 3.

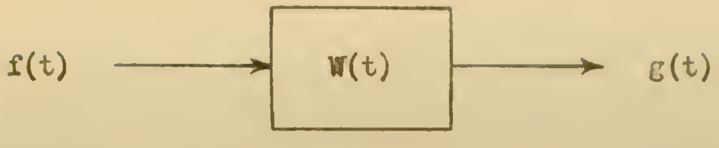


Figure 3. Time Domain Filtering.

The following correspondence exists between the time and frequency domain filters.

$$F(s) = \mathcal{L}\left[f(t)\right] \tag{6}$$

$$H(s) = \mathcal{L}\left[W(t)\right] \tag{7}$$

This duality between the time and frequency domains is further shown by the Gardner and Barnes[6] complex multiplication Laplace transform pairs.

$$f_1(t) * f_2(t) \longleftrightarrow F_1(s)\ F_2(s)$$

A technique utilizing both the time and frequency domains will be used later. It consists of finding the appropriate weighting function, W(t), and then taking its Laplace transform to obtain H(s). In this case the properties of the filter are more readily found in the time domain. The switch to the frequency domain is employed so that conventional methods of network synthesis may be used.

5. Numerical Least Squares Curve Fitting.

Related to the discussion of the second section of this chapter is the static case of least squares curve fitting. Scarborough[26] has discussed both the basis for the method and the steps necessary to carry out the numerical computation. It is based on the Normal Law of Error,

the Principle of Least Squares, and the Law of Error for Residuals. Consider the function, P, which is the sum of several functions, $f_1(x)$, $f_2(x)$, . . . $f_n(x)$.

$$P = b_1 f_1(x) + b_2 f_2(x) + \dots + b_n f_n(x) \tag{8}$$

Pairs of observed values, (x_1, P_1), (x_2, P_2), . . . (x_m, P_m), are obtained from a set of m measurements of equal precision. It is desired to find the values of the coefficients, b_1, b_2, . . . b_n, such that a graph of P will come as near as possible to each of the m points. A set of m residual equations is obtained by substituting each of the pairs of values in turn into equation (8).

$$v_1 = b_1f_1(x_1) + b_2f_2(x_1) + \dots + b_nf_n(x_1) - P_1 \tag{9}$$

$$v_2 = b_1f_1(x_2) + b_2f_2(x_2) + \dots + b_nf_n(x_2) - P_2 \tag{10}$$

. . . .

$$v_m = b_1f_1(x_m) + b_2f_2(x_m) + \dots + b_nf_n(x_m) - P_m \tag{11}$$

The best values of the unknown coefficients are those which make the sum of the squares of the residuals a minimum.

$$v_1^2 + v_2^2 + \dots + v_m^2 = f(b_1, b_2, \dots b_n) = \text{minimum} \tag{12}$$

To make equation (12) a maximum or minimum, the partial derivatives of $f(b_1, b_2, \dots b_n)$ with respect to each of the coefficients shall be zero. The values of the residuals, v_1, v_2, . . . v_m, given in equations (9) through (11) are substituted into equation (12); the partial differ-

entiations are performed with respect to $b_1, b_2, \ldots b_n$; and the n resulting equations are equated to zero. After dividing through by 2, a set of simultaneous equations, called normal equations, are obtained. They may be solved by ordinary algebra.

$$\begin{aligned} & f_1(x_1)\left[b_1 f_1(x_1) + b_2 f_2(x_1) + \ldots + b_n f_n(x_1) - P_1\right] \\ & \quad + f_1(x_2)\left[b_1 f_1(x_2) + b_2 f_2(x_2) + \ldots + b_n f_n(x_2) - P_2\right] + \ldots \\ & \quad + f_1(x_m)\left[b_1 f_1(x_m) + b_2 f_2(x_m) + \ldots + b_n f_n(x_m) - P_m\right] = 0 \end{aligned} \tag{13}$$

$$\begin{aligned} & f_2(x_1)\left[b_1 f_1(x_1) + b_2 f_2(x_1) + \ldots + b_n f_n(x_1) - P_1\right] \\ & \quad + f_2(x_2)\left[b_1 f_1(x_2) + b_2 f_2(x_2) + \ldots + b_n f_n(x_2) - P_2\right] + \ldots \\ & \quad + f_2(x_m)\left[b_1 f_1(x_m) + b_2 f_2(x_m) + \ldots + b_n f_n(x_m) - P_m\right] = 0 \end{aligned} \tag{14}$$

. . . .

$$\begin{aligned} & f_n(x_1)\left[b_1 f_1(x_1) + b_2 f_2(x_1) + \ldots + b_n f_n(x_1) - P_1\right] \\ & \quad + f_n(x_2)\left[b_1 f_1(x_2) + b_2 f_2(x_2) + \ldots + b_n f_n(x_2) - P_2\right] + \ldots \\ & \quad + f_n(x_m)\left[b_1 f_1(x_m) + b_2 f_2(x_m) + \ldots + b_n f_n(x_m) - P_m\right] = 0 \end{aligned} \tag{15}$$

If each unknown coefficient is to be uniquely determined, the number of independent sets of observations must be equal to or greater than the number of coefficients. By increasing the number of observations, a better fit is obtained.

When measurements are of unequal precision, the least square method is extended by having the sum of the weighted squares of the residuals a minimum.

$$\sum_{j=1}^{m} w_j v_j^2 = w_1 v_1^2 + w_2 v_2^2 + \ldots + w_m v_m^2 = \text{minimum} \tag{16}$$

The weights, w_1, w_2, . . . w_m, depend upon the precision with which each pair of measurements is made, the more precise measurements receiving greater weights.

6. The Finite Fourier Series as a Minimum Mean-Square Approximation.

As an introduction to a technique which will be utilized later, the least square property of the finite Fourier series will be shown. Let a function, f(t), which is piecewise continuous and defined from $c \leqq t \leqq c + 2\pi$, be approximated by the function g(t).

$$g(t) = a_0 + a_1 \cos t + b_1 \sin t + a_2 \cos 2t + b_2 \sin 2t + \dots + a_n \cos nt + b_n \sin nt \qquad (17)$$

$$g(t) = \sum_{k=0}^{n} a_k \cos kt + b_k \sin kt \qquad (18)$$

It is desired to choose a_k and b_k such that the mean-square error, e(t), over the interval for which f(t) is defined is a minimum.

$$e(t) = \frac{1}{2\pi}\int_{c}^{c+2\pi} \left[f(t) - g(t)\right]^2 dt = \text{minimum} \qquad (19)$$

For a minimum, the partial derivatives of e(t) with respect to a_k and b_k should be zero. The value of g(t) given by equation (18) is substituted into equation (19). The partial derivative with respect to a_k may be taken under the integral sign and the resulting expression equated to zero.

$$\frac{1}{2\pi}\int_{c}^{c+2\pi} 2\left[f(t) - \sum_{k=0}^{n}(a_k \cos kt + b_k \sin kt)\right]\left[-\sum_{j=0}^{n} \cos jt\right] dt = 0 \qquad (20)$$

$$\frac{1}{\pi}\int_{c}^{c+2\pi} \sum_{k=0}^{n} \sum_{j=0}^{n} \Big[a_k \cos kt \ \cos jt + b_k \sin kt \ \cos jt \Big] dt = \frac{1}{\pi}\int_{c}^{c+2\pi} f(t) \sum_{j=0}^{n} \cos jt \ dt \qquad (21)$$

To evaluate the left side of equation (21), certain orthogonal properties of the sine and cosine should be recalled.

$$\int_{c}^{c+2\pi} \cos nx \ \cos mx \ dx = 0 \qquad \text{for } n \neq m \qquad (22)$$

$$\int_{c}^{c+2\pi} \cos nx \ \cos mx \ dx = \pi \qquad \text{for } n = m \qquad (23)$$

$$\int_{c}^{c+2\pi} \sin nx \ \cos mx \ dx = 0 \qquad \text{for all } n \text{ and } m \qquad (24)$$

The only contribution to the left side of equation (21) occurs when k and j are equal. Equation (21) may now be written as

$$\sum_{k=0}^{n} a_k = \frac{1}{\pi}\sum_{k=0}^{n} \int_{c}^{c+2\pi} f(t) \cos kt \ dt \qquad (25)$$

As a result of the rearrangement of equation (20) to get equation (21), there is an equality between terms which arise from substituting the same value of k on each side of equation (25). Hence the general expression for a_k is

$$a_k = \frac{1}{\pi}\int_{c}^{c+2\pi} f(t) \cos kt \ dt \qquad \text{where } k = 0, 1, \ldots n \qquad (26)$$

This is recognized as the general expression for the coefficients of the cosine terms in the Fourier series expansion. A similar process of taking the partial derivative with respect to b_k will yield the general expression for the coefficients of the sine terms in the Fourier series expansion.

The remaining question to be settled is whether the values of a_k and b_k just obtained represent a maxima, minima, or saddle point. This is accomplished by finding the values of the three second partial derivatives represented by A, B, and C at the point (a_k, b_k). The differentiations may again be carried out under the integral sign.

$$A = \frac{\partial^2 e(t)}{\partial a_k^2}(a_k, b_k) = \frac{1}{\pi}\int_c^{c+2\pi} \sum_{k=0}^{n} \sum_{j=0}^{n} \cos kt \quad \cos jt \quad dt = 1 \tag{27}$$

$$B = \frac{\partial^2 e(t)}{\partial a_k \partial b_k}(a_k, b_k) = \frac{1}{\pi}\int_c^{c+2\pi} 0 \quad dt = 0 \tag{28}$$

$$C = \frac{\partial^2 e(t)}{\partial b_k^2}(a_k, b_k) = \frac{1}{\pi}\int_c^{c+2\pi} \sum_{k=0}^{n} \sum_{j=0}^{n} \sin kt \quad \sin jt \quad dt = 1 \tag{29}$$

The conditions for a relative minima of a function of two variables are

$$B^2 - AC < 0 \tag{30}$$

$$A + C > 0 \tag{31}$$

Both conditions are met in this case.

Thus if f(t) be piecewise continuous for $c \leqq t \leqq c + 2\pi$, then the coefficients of the partial sum of the Fourier series of f(t) are precisely those among all coefficients of the function, g(t), which render the mean-square error a minimum.

The foregoing proof was suggested by a problem in Hildebrand[13]. Both Guillemin[9] and Kaplan[16] prove the same result using different approaches.

CHAPTER II AN EXTENSION OF LEAST SQUARES PREDICTION THEORY

1. Basic Assumptions.

The discussion of this chapter will be based upon the general problem as stated in Chapter I. The received, observed, or measured signal, R(t), which is the input to the filter, consists of a true signal, P(t), and noise, N(t). The true signal, P(t), will be assumed to consist of n completely foreknown functions of time, $f_1(t)$, $f_2(t)$, . . . $f_n(t)$, each multiplied by a constant coefficient, a_1, a_2, . . . a_n, which is not a function of time.

$$P(t) = a_1 f_1(t) + a_2 f_2(t) + \dots + a_n f_n(t) \tag{32}$$

$$P(t) = \sum_{h=1}^{n} a_h f_h(t) \tag{33}$$

Such functions as $e^{a_1 t}$ and $\sin a_2 t$ are not considered since they are in effect undetermined functions.

From a foreknowledge of the form of P(t) and our operations on R(t), we attempt to obtain a closer estimate of P(t), called Q(t), such that Q(t) will differ from R(t) by the least amount in some manner or other. The "best" estimate, Q(t), will be chosen to have the same form as P(t), that is

$$Q(t) = \sum_{k=1}^{n} b_k g_k(t) \tag{34}$$

The functions, $g_k(t)$, in the expression for Q(t) are closely related to the functions, $f_h(t)$, in the expression for P(t) and in many cases may be identical. The exact restrictions imposed upon the functions making

up P(t) and Q(t) will be discussed in the next section. No assumption will be made about the type of noise, N(t). It will be assumed that a reasonable criterion of performance is one in which the time weighted squared error is minimized as illustrated in section three of this chapter. This minimization is brought about by the way in which the undetermined constants, b_1, b_2, . . . b_n, are selected.

In order to simplify the physical realization of the filters, they will be assumed to be constructed of linear elements which do not vary with time. The limited scope of this paper will not permit a thorough discussion of a broader class of filters which may be constructed by permitting the size of the filter elements to vary with time.

In general, we assume the function, P(t), to be completely foreknown. There are exceptions, however, which are amenable to treatment even if there is some degree of uncertainty as to the form of P(t). This occurs when the function is unknown in the sense that it is one of several known possible functions. The filter may be designed to take care of all the known possibilities. As an example, suppose that P(t) is known to be either a_1, $a_2\,e^t$, or $a_3\,e^{2t}$. Then the actual signal for which the filter is designed might be

$$Q(t) \;=\; b_1 \;+\; b_2\,e^t \;+\; b_3\,e^{2t} \qquad (35)$$

The price paid for this uncertainty as to the exact function present is an increase in the noise output of the filter. This is so because the inclusion of additional functions will in general require the filter to have a greater frequency response than that required for the one true signal alone. The increase in filter noise output limits the extent to

which this method can be carried. Another case is that of the undetermined function $a_1 \sin (wt + a_2)$. As it stands, it cannot be treated. However, by putting it in the equivalent form, $b_1 \sin wt + b_2 \cos wt$, it may be considered. In this case there is no loss of generality nor increase in the noise output.

2. Applicable Functions.

The only assumption which will be made about the type of functions comprising Q(t) is that they form a closed set under a linear translation in time, that is, no new functions will be generated. This restriction results from the fact that, for simplicity, the filters are considered to be invariant with time. In going from a static case of least squares curve fitting to the dynamic situation, no time reference is provided the filter. Given a sinusoidal type wave that has been going on for an indefinitely long time, one cannot say whether the wave is a sine or a cosine unless some reference time is chosen. However, no matter what reference is chosen, the wave can always be described as some linear combination of cos wt and sin wt. To understand what this restriction implies, consider the functions $b_1 \sin wt$ and $b_2 t^2$ under a translation in time of duration c.

$$b_1 \sin w(t + c) = b_3 \sin wt + b_4 \cos wt \tag{36}$$

$$b_2 (t + c)^2 = b_5 + b_6 t + b_7 t^2 \tag{37}$$

In both cases, new functions resulted from the translations in time. Now if the expressions on the right sides of equations (36) and (37)

undergo a further translation in time, the translations will merely change the coefficients without generating any new functions. Thus the right sides satisfy the requirement for functions which make up Q(t). It may also be observed that functions of the form $b_1\ t^{1.4}$ and $\dfrac{1}{b_1 + b_2\ t + b_3\ t^2}$ can never satisfy this restriction.

The next question requiring an answer is, "What functions remain a closed set under a linear transformation in time?" As an illustration, consider the case in which there are two functions involved.

$$Q(t) = b_1\ g_1(t) + b_2\ g_2(t) \tag{38}$$

The linear transformation may be written in matrix form.

$$\begin{bmatrix} g_1(t + \Delta t) \\ g_2(t + \Delta t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} \tag{39}$$

But A, B, C, and D must be chosen so that when Δt equals zero

$$\begin{bmatrix} g_1(t + 0) \\ g_2(t + 0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} \tag{40}$$

This condition is satisfied if

$$\begin{bmatrix} g_1(t + \Delta t) \\ g_2(t + \Delta t) \end{bmatrix} = \begin{bmatrix} 1 + a\ \Delta t & b\ \Delta t \\ c\ \Delta t & 1 + d\ \Delta t \end{bmatrix} \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} \tag{41}$$

where a, b, c, and d are arbitrary constants. Subtract the matrix $\begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}$ from both sides of the matrix equation (41) and divide by Δt.

$$\begin{bmatrix} \dfrac{g_1(t+\Delta t) - g_1(t)}{\Delta t} \\ \dfrac{g_2(t+\Delta t) - g_2(t)}{\Delta t} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} \tag{42}$$

Since this must hold for all Δt, let Δt approach the limit zero. The resulting derivative is denoted by a prime.

$$\begin{bmatrix} g_1'(t) \\ g_2'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} \tag{43}$$

Now taking the second derivatives with respect to time,

$$\begin{bmatrix} g_1''(t) \\ g_2''(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} g_1'(t) \\ g_2'(t) \end{bmatrix} \tag{44}$$

The matrix equations (43) and (44) may now be used to find similar second order differential equations in $g_1(t)$ and $g_2(t)$. These equations take the form

$$g''(t) + C_1 g'(t) + C_2 g(t) = 0 \quad \text{where } C_1 \text{ and } C_2 \text{ are constants} \tag{45}$$

By using similar methods, it is evident that if Q(t) consists of n functions, these functions must be solutions of the reduced equation of an n th order linear differential equation with constant coefficients. This restriction is not as serious as it might first appear since a wide variety of useful functions belong in this group. Included are exponentials, polynomials, sinusoids, and functions formed by adding or multiplying these types together. While the functions, f(t), that form

the true signal, $P(t)$, may be considered to be any one solution of the differential equation such as $a_1 \sin wt$, $Q(t)$ must consist of the corresponding complete set of functions, $b_1 \sin wt + b_2 \cos wt$. Similarly, if $f(t)$ has a polynomial term such as $a_2 t^2$, $g(t)$ must contain the corresponding complete polynomial, $b_3 + b_4 t + b_5 t^2$.

3. Time Weighted Least Squares Curve Fitting.

The process which will be used to obtain a time weighted least squares fit is illustrated in Figure 4. The top sketch shows the noisy signal, $R(t)$, and the best estimate of the true signal, $Q(t)$. The difference between the two signals is the residual error, $R(t) - Q(t)$. In the middle of the figure is the square of this residual error, $\left[R(t) - Q(t)\right]^2$. In order to allow some latitude in the construction of the filter, this squared error is weighted over the past in some arbitrary manner by multiplying the square of the residual error by a weighting function of time, $M(t)$. The result of the multiplication is to produce a time weighted squared residual error curve as shown by the crosshatched area of the bottom diagram. The area underneath this curve is given by

$$A = \int_{-\infty}^{0} \left[R(t) - Q(t)\right]^2 M(t)\, dt \tag{46}$$

It is precisely this crosshatched area which will be minimized.

The squared error weighting function, $M(t)$, may be thought of as the filter's memory of the squared error. The mathematical restriction on $M(t)$ is that it be chosen so that the area integral is finite. There is, however, another practical restriction on $M(t)$. Since one must wait

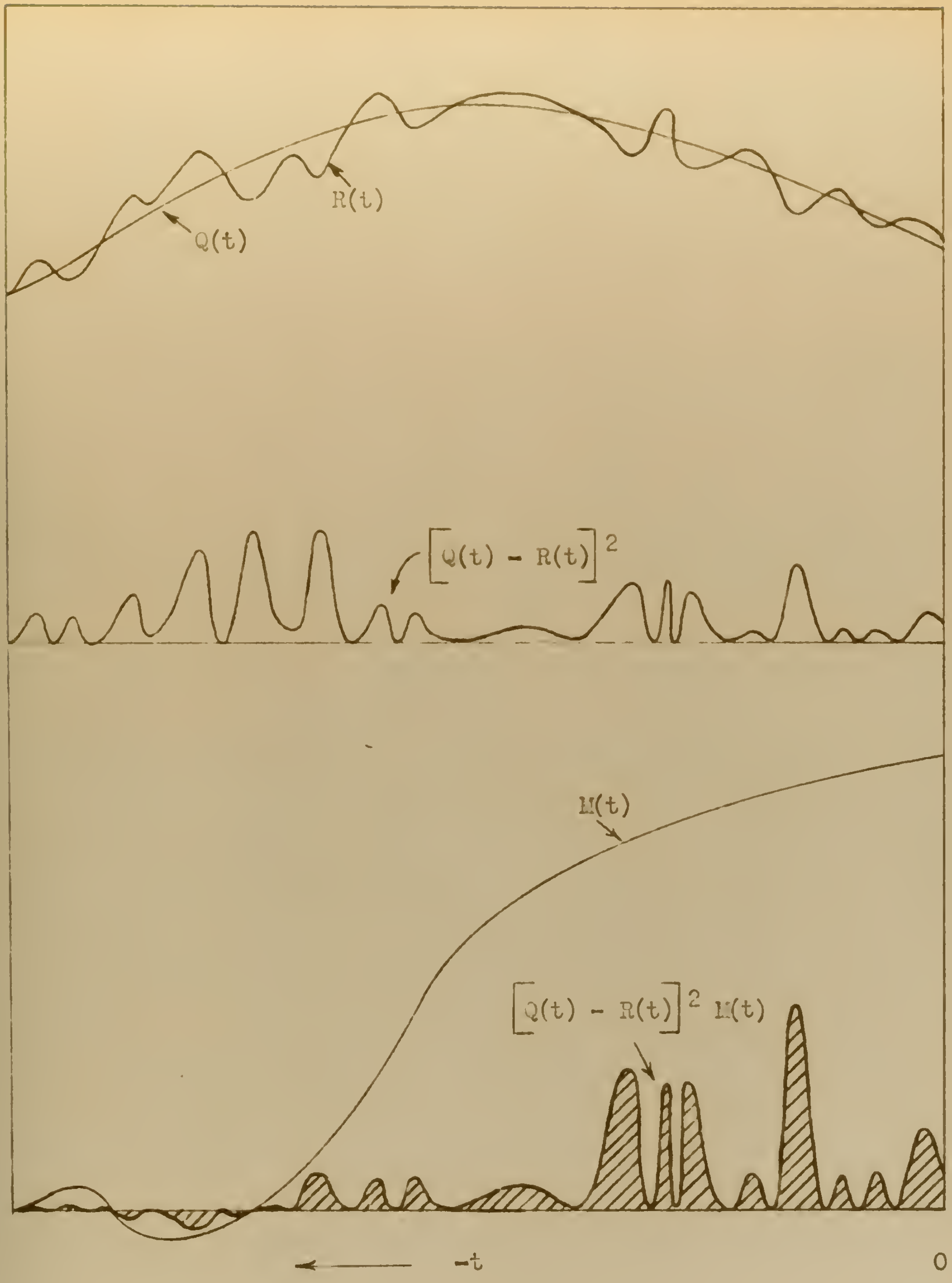


Figure 4. Time Weighted Least Squares Curve Fitting.

until the smoothing time, T, has been completed or until the filter has had time to settle before it provides an "optimum" output, the memory of the filter cannot extend too far into the past. M(t) may be a compromise between a long smoothing time which makes use of more information about the signal and a short time which would provide a quicker decision as to the nature of the signal.

Probably the three most important forms of M(t) are the rectangular, exponential, and damped sinusoidal types shown in Figure 5. In the rectangular type, the filter weights all squared errors from a certain fixed time in the past, T, up to the present time equally. This corresponds to the weighting discussed by Zadeh and Ragazzini[30] in which

$$M(t) = u(-t) - u(-t-T) \quad \text{where } u(t) \text{ is the unit step function} \qquad (47)$$

The abrupt change in weighting results in filters which contain a delay line type element. Another useful type filter is one which forgets what has gone on in the past exponentially. Exponential weighting is defined by

$$M(t) = e^{t/T}\, u(-t) \qquad (48)$$

This type of memory is very promising in that it corresponds to the natural damping found in most physical systems with dissipative elements, and hence holds forth the hope that the filters may be realized with lumped constant elements. The third type of weighting defined by

$$M(t) = e^{t/T}\,(\cos \pi t - \sin \pi t)\, u(-t) \qquad (49)$$

is an attempt to approximate the rectangular filter. The output of this filter would tend to have a transient damped oscillation and also holds

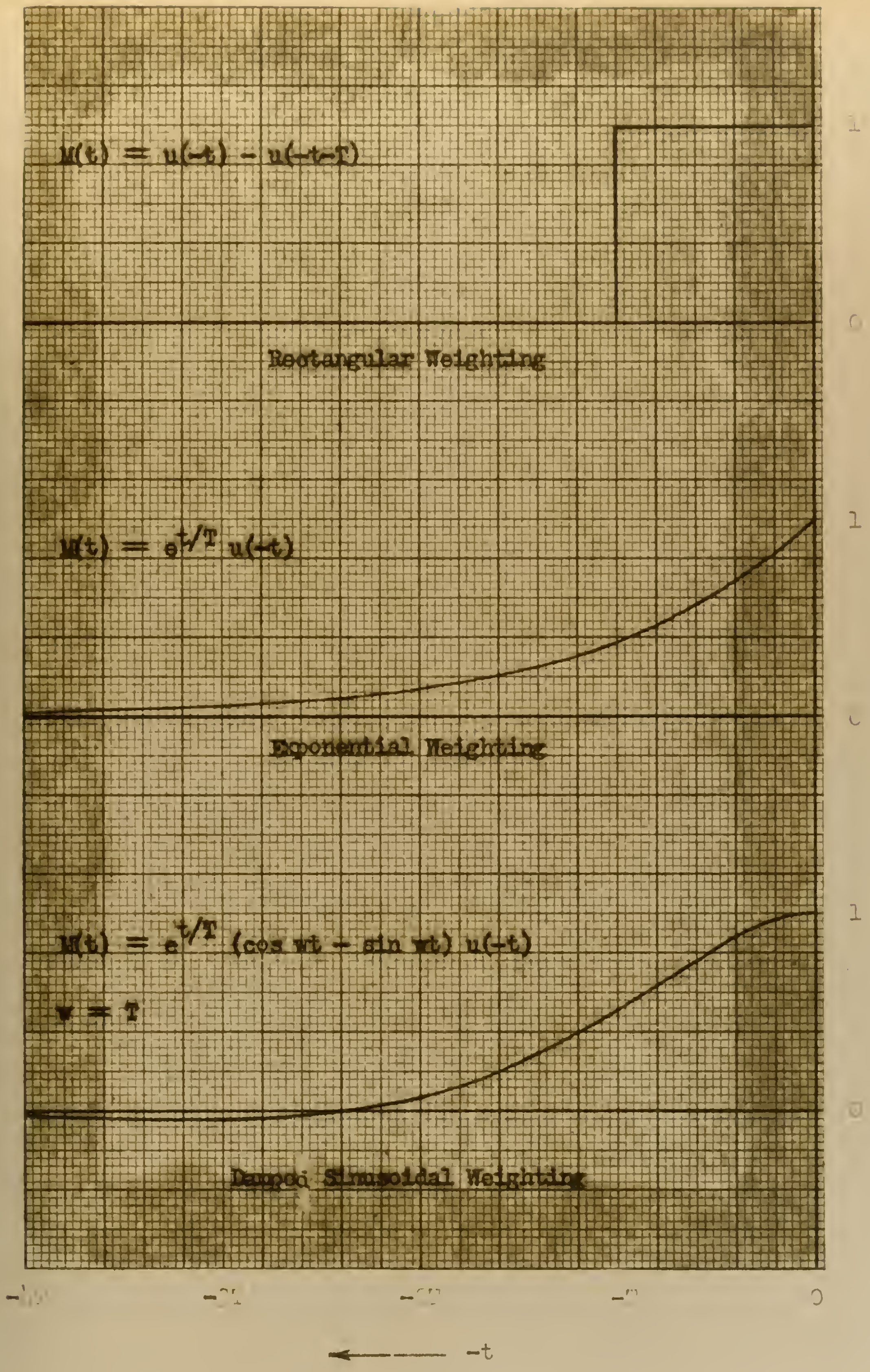


Figure [illegible]. Typical Squared Error Weighting Functions.

forth the hope of realization with lumped constant elements. Other types of weighting which might find application under certain circumstances include t $e^{t/T}$, $t^2 e^{t/T}$, and $e^{t/T}$ sin wt.

4. Formulation of the Weighting Function.

If the expression for Q(t) given in equation (34) is substituted into equation (46), the expression for the crosshatched area in Figure 4 becomes

$$A = \int_{-\infty}^{0} \left[R(t) - \sum_{k=1}^{n} b_k \, g_k(t) \right]^2 M(t) \, dt \qquad (50)$$

The conditions for a minimum are that the partial derivatives of the expression for the area with respect to the coefficients b_1, b_2, . . . b_n equal zero. This results in a set of n simultaneous linear equations.

$$\frac{\partial A}{\partial b_k} = 0 \qquad \text{where } k = 1, 2, \ldots n \qquad (51)$$

In order to carry out these differentiations under the integral sign, a form of Leibnitz's rule similar to that given by Kaplan[17] is now stated.

Let f(x, t) be continuous and have a continuous derivative $\partial f/\partial x$ in a domain of the xt plane which includes the rectangle $a \leqq x \leqq b$, $t_1 \leqq t \leqq t_2$. Then for $a < x < b$

$$\frac{d}{dx} \int_{t_1}^{t_2} f(x, t) \, dt = \int_{t_1}^{t_2} \frac{\partial f}{\partial x} (x, t) \, dt \qquad (52)$$

The partial derivative with respect to the general coefficient, b_k, will be taken under the integral sign.

$$\frac{\partial A}{\partial b_k} = \int_{-\infty}^{0} \frac{\partial}{\partial b_k}\left[R(t) - \sum_{k=1}^{n} b_k\, g_k(t)\right]^2 M(t)\, dt \tag{53}$$

$$\frac{\partial A}{\partial b_k} = -2\int_{-\infty}^{0} M(t)\left\{\sum_{j=1}^{n} g_j(t)\left[R(t) - \sum_{k=1}^{n} b_k\, g_k(t)\right]\right\} dt \tag{54}$$

The change of subscript from k to j was made to indicate that the summations are distinct. Setting the partial derivative equal to zero and rearranging the resulting equation

$$\int_{-\infty}^{0} R(t)\sum_{j=1}^{n} g_j(t)\, M(t)\, dt = \int_{-\infty}^{0}\left[\sum_{k=1}^{n} b_k\, g_k(t)\right]\sum_{j'=j} g_{j'}(t)\, M(t)\, dt \tag{55}$$

If two new symbols are defined as follows,

$$I_j = \int_{-\infty}^{0} R(t)\, g_j(t)\, M(t)\, dt \tag{56}$$

$$B_{jk} = B_{kj} = \int_{-\infty}^{0} g_k(t)\, g_j(t)\, M(t)\, dt \tag{57}$$

then B_{jk} can be evaluated since $g_k(t)$, $g_j(t)$, and $M(t)$ are all known, and the set of n simultaneous equations defined by equation (55) take the simpler form

$$\sum_{j=1}^{n} I_j = \sum_{k=1}^{n}\sum_{j'=j} B_{j'k}\, b_k \tag{58}$$

In matrix form these equations become

$$\begin{bmatrix} I_1 \\ I_2 \\ \cdot\ \cdot \\ I_n \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \cdot\ \cdot \\ b_n \end{bmatrix}. \tag{59}$$

Let C_{ki} be the inverse matrix of B_{jk}, that is

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \qquad (60)$$

Premultiplying both sides of equation (59) by the C_{ki} matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \cdots \\ I_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix} \qquad (61)$$

or, in the form of a summation,

$$b_k = \sum_{i=1}^{n} C_{ki}\, I_i \qquad (62)$$

Substituting from equation (62) into equation (34)

$$Q(t) = \sum_{k=1}^{n} \sum_{i=1}^{n} I_i\, C_{ki}\, g_k(t) \qquad (63)$$

To avoid confusion with the actual variable, t, the dummy variable of integration in equation (56) is changed from t to x. The resulting value of I_i is substituted into equation (63).

$$Q(t) = \sum_{k=1}^{n} \sum_{i=1}^{n} \left[\int_{-\infty}^{0} R(x)\, g_i(x)\, M(x)\, dx \right] C_{ki}\, g_k(t) \qquad (64)$$

Rearranging equation (64) so that all terms are under the integral sign,

$$Q(t) = \int_{-\infty}^{0} \sum_{k=1}^{n} \sum_{i=1}^{n} R(x)\, g_i(x)\, C_{ki}\, g_k(t)\, M(x)\, dx \qquad (65)$$

Let C_{ki} be the inverse matrix of $B_{j'k}$, that is

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{60}$$

Premultiplying both sides of equation (59) by the C_{ki} matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \cdot\,\cdot \\ I_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot\,\cdot \\ b_n \end{bmatrix} \tag{61}$$

or, in the form of a summation,

$$b_k = \sum_{i=1}^{n} C_{ki}\, I_i \tag{62}$$

Substituting from equation (62) into equation (34)

$$Q(t) = \sum_{k=1}^{n} \sum_{i=1}^{n} I_i\, C_{ki}\, g_k(t) \tag{63}$$

To avoid confusion with the actual variable, t, the dummy variable of integration in equation (56) is changed from t to x. The resulting value of I_i is substituted into equation (63).

$$Q(t) = \sum_{k=1}^{n} \sum_{i=1}^{n} \left[\int_{-\infty}^{0} R(x)\, g_i(x)\, M(x)\, dx \right] C_{ki}\, g_k(t) \tag{64}$$

Rearranging equation (64) so that all terms are under the integral sign,

$$Q(t) = \int_{-\infty}^{0} \sum_{k=1}^{n} \sum_{i=1}^{n} R(x)\, g_i(x)\, C_{ki}\, g_k(t)\, M(x)\, dx \tag{65}$$

Consider that Q(t) is the output of a filter with an input, R(t). Referring to Figure 4, it is seen that Q(0) is the best estimate of the true signal at the present time. If t is positive, the value of Q(t) is an estimate of a future value of the true signal. Thus the quantity t is the prediction time. For the fixed filters under consideration, let t be a fixed prediction time designated by the symbol θ. Equation (65) may now be interpreted as a convolution integral of the form shown by equation (66).

$$Q(\theta) = \int_{-\infty}^{0} W(\theta,\ 0 - x)\ R(x)\ dx \qquad (66)$$

A comparison of equations (65) and (66) indicates that the weighting function of the filter must be

$$W(\theta,\ 0 - x) = \sum_{k=1}^{n} \sum_{i=1}^{n} g_i(x)\ C_{ki}\ g_k(\theta)\ M(x) \qquad (67)$$

Making the substitution

$$t = 0 - x \qquad (68)$$

the weighting function of the filter becomes

$$W(\theta,\ t) = \sum_{k=1}^{n} \sum_{i=1}^{n} g_i(-t)\ C_{ki}\ g_k(\theta)\ M(-t)\ u(t) \qquad (69).$$

where u(t) is the unit step function. In matrix form this becomes

$$W(\theta,\ t) =$$

$$\begin{bmatrix} g_1(-t), & g_2(-t), & \cdots & g_n(-t) \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} g_1(\theta) \\ g_2(\theta) \\ \cdots \\ g_n(\theta) \end{bmatrix} M(-t)\ u(t) \qquad (70)$$

Since most network synthesis is done in the frequency domain, it is convenient to represent the filter by its complex frequency transfer function, H(s), the Laplace transform of the weighting function.

$$H(s) = \int_0^{\infty} W(\theta,\ t)\ e^{-st}\ dt \tag{71}$$

To obtain smooth derivatives or integrals, H(s) is either multiplied or divided by the variable s.

5. Steady State Filter Response in the Absence of Noise.

If the filter weighting functions found in the preceeding section are valid, a proper input to the filter in the absence of noise should produce a steady state output with no error. Since the noise, N(t), is zero, the input, R(t), is equal to the true signal, P(t). From equations (1) and (33), the input may be expressed as

$$R(t) = \sum_{h=1}^{n} a_h\ f_h(t) \tag{72}$$

The filter weighting function is given by equation (69) as

$$W(\theta,\ t) = \sum_{k=1}^{n} \sum_{i=1}^{n} g_i(-t)\ C_{ki}\ g_k(\theta)\ M(-t)\ u(t) \tag{73}$$

From equation (5), the steady state output of the filter is

$$Q(t) = \int_0^{\infty} W(\tau)\ R(t-\tau)\ d\tau = \int_{\infty}^{0} W(-\tau)\ R(t+\tau)\ d\tau \tag{74}$$

Substituting the expressions for the input and weighting function into the equation for the steady state output, one obtains

$$Q(t) = \int_{-\infty}^{0} \sum_{k=1}^{n} \sum_{i=1}^{n} g_i(\tau)\ C_{ki}\ g_k(\theta)\ M(\tau) \left[\sum_{h=1}^{n} a_h\ f_h(t+\tau) \right] d\tau \tag{75}$$

Assume that $\sum_{h=1}^{n} f_h(t + \tau)$ may be expressed as

$$\sum_{h=1}^{n} f_h(t + \tau) = \sum_{h'=h} \sum_{p=1}^{n} D_{h'p}(t)\, g_p(\tau) \tag{76}$$

If equation (76) is substituted into equation (75), then

$$Q(t) = \sum_{k=1}^{n} \sum_{h=1}^{n} \sum_{i=1}^{n} \sum_{p=1}^{n} C_{ki}\, g_k(\theta)\, a_h\, D_{hp}(t) \int_{-\infty}^{0} g_i(\tau)\, g_p(\tau)\, M(\tau)\, d\tau \tag{77}$$

From equation (57) the integral B_{ip} may be defined as

$$B_{ip} = \int_{-\infty}^{0} g_i(\tau)\, g_p(\tau)\, M(\tau)\, d\tau \tag{78}$$

This will reduce equation (77) to the form

$$Q(t) = \sum_{k=1}^{n} \sum_{h=1}^{n} \sum_{i=1}^{n} \sum_{p=1}^{n} B_{ip}\, C_{ki}\, g_k(\theta)\, a_h\, D_{hp}(t) \tag{79}$$

The relation between B_{ip} and C_{ki} is

$$\begin{aligned} \sum B_{ip}\, C_{ki} &= 1 \qquad \text{when } p = k \\ &= 0 \qquad \text{when } p \neq k \end{aligned} \tag{80}$$

Equation (79) may now be reduced to

$$Q(t) = \sum_{k=1}^{n} \sum_{h=1}^{n} g_k(\theta)\, a_h\, D_{hk}(t) \tag{81}$$

By comparison with equation (76)

$$\sum_{k=1}^{n} \sum_{h=1}^{n} D_{hk}(t)\, g_k(\theta) = \sum_{h=1}^{n} f_h(t + \theta) \tag{82}$$

so that Q(t) can be expressed as

$$Q(t) = \sum_{h=1}^{n} a_h\, f_h(t + \theta) \tag{83}$$

This expression may now be compared with equation (72) to give

$$Q(t) = R(t + \theta) \qquad (84)$$

Thus the steady state output of the filter equals the filter input displaced by the amount of the prediction time, θ.

The only assumption made in this section was that defined by equation (76). It leads to a set of equations very similar to the matrix set given in equation (39). This assumption corresponds to the requirement that the output of the filter have the same form as the input except for a possible translation in time. One may again conclude that the functions which are amenable to treatment are the solutions of the reduced equation of an n th order linear differential equation with constant coefficients.

CHAPTER III REALIZATION OF SMOOTHING AND PREDICTING FILTERS

1. Methods Available for Construction.

The formulation of the weighting function of Chapter II did not insure that every filter was physically realizable using lumped constant elements. It did, however, provide such a wide variety of squared error memory functions, M(t), that a high probability exists that the filters may be constructed by one of several methods.

When a rectangular error memory function is used, it is often convenient to synthesize the transfer functions with operational amplifiers such as those used in analog computers. This is especially true if the signal under consideration is composed of low frequency components only. Since the rectangular error memory function leads to a delay line element, the operational amplifiers have the advantage of providing isolation and a means of summing the undelayed and delayed signals. While conventional delay lines could be used for shorter smoothing times, it would probably be necessary to resort to a continuous tape or drum arrangement such as the one shown in Figure 6 for the longer smoothing times. The actual synthesis of these filters will not be discussed in detail since the needed information may be found in books on analog computers such as the one by Korn and Korn[20].

If an exponential or damped sinusoidal error memory function is used, it is often possible to synthesize the filter as a two terminal driving point impedance, Z(s). The input to the filter is a current which could be supplied from a pentode while the output is the voltage across the terminals. A driving point impedance must be a positive real

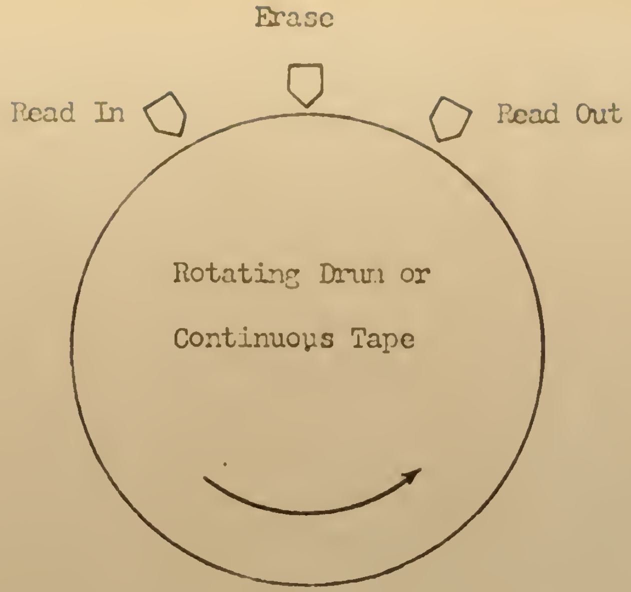


Figure 6. Magnetic Tape or Drum Delay Element.

function. The well known conditions for a rational function to be a positive real function are

Z(s) is real when s is real.

Z(s) is analytic in the right half plane

Poles on the boundary are simple with positive, real residues.

$\mathrm{Re}\left[Z(jw)\right] \geq 0 \quad$ when $\mathrm{Re}\left[s\right] \geq 0$

If the transfer function meets these conditions, the synthesis of the filters as driving point impedances may often be done by inspection, and can always be accomplished by the methods of either Brune[4] or Bott and Duffin[3,10].

A second method for synthesizing the transfer function is available when the exponential or damped sinusoidal error memory function is used. The transfer function, H(s), may be considered to be the ratio of the open circuit output voltage of a two-terminal pair network to the input voltage. Kahal[15] has shown the conditions for realizing the transfer

function to within a constant factor using a finite number of lumped constant, linear elements. For the symmetrical lattice network these conditions are

H(s) is analytic in the right half s plane.

Poles on the boundary are simple with residues having a zero real part.

It is evident that a wider class of functions may be treated by this method than in the case of driving point impedances. However, there is a practical problem which arises from the fact that no common ground exists between the input and output terminals. This difficulty will not be present if the synthesis is carried out as a ladder network. Kahal[15] has shown that one additional condition besides those for the symmetrical lattice must be satisfied to realize the transfer function to within a constant factor as a ladder network. This condition is that H(s) must have no zeros in the right half plane. A major problem in the actual synthesis of the filters is usually the factorization of the transfer function. Two methods for carrying out the actual synthesis of the filter are those of Darlington[5], and Weinberg[27, 28].

2. Filter Design Procedure.

Before proceeding with examples of specific filters, it would be well to list the general steps to be followed in the design of any filter. Usually the function(s) to be smoothed, the prediction time, and the functional desired are determined by the requirements of the problem.

Step 1. From the known function(s) to be smoothed, find the complete set of functions which form a closed set under a translation in time.

If the smoothed integral of a polynomial is desired, the filter must be designed for the next higher order polynomial. Conversely, the smoothed derivative of a polynomial will be designed for the next lower order polynomial.

Step 2. Choose a squared error memory function, M(t).

Step 3. Find the elements of the B_{jk} matrix in accordance with equation (57).

Step 4. Find the C_{ki} matrix which is the inverse of the B_{jk} matrix. Methods for obtaining the inverse matrix are given in the mathematics texts by Pipes[24] and Guillemin[11].

Step 5. Find the numerical values of the functions of step 1 at the prediction time, θ.

Step 6. Determine the filter weighting function as given by equation (70).

Step 7. Take the Laplace transform of the weighting function. Multiply the result by s if a derivative is desired and divide it by s if an integral is desired.

Step 8. Use the complex frequency transfer function found in step 7 as the basis for the synthesis of the network.

3. Smoothing and Prediction of an Exponential Function.

The only functions which are solutions of the first order linear differential equation with constant coefficients

$$f' + C_1 f = 0 \qquad \text{where } C_1 \text{ is a constant} \tag{85}$$

are of the type $a_1 e^{mt}$. The steps of the preceeding section will be

followed to find filters which both smooth and predict for this function.

For the first case it is assumed that rectangular weighting of the squared error is to be used to smooth the noisy function $a_1 e^{mt}$. The best estimate of the function itself is desired with a prediction time of θ seconds and a smoothing time of T seconds.

Step 1. $$g_1 = e^{mt} \tag{86}$$

Step 2. $$M(t) = u(-t) - u(-t-T) \tag{87}$$

Step 3. $$B_{11} = \int_{-T}^{0} e^{2mt}\, dt = \frac{1}{2m}\left[1 - e^{-2m}\right] \tag{88}$$

Step 4. $$C_{11} = \frac{2m}{1 - e^{-2m}} \tag{89}$$

Step 5. $$g_1(\theta) = e^{m\theta} \tag{90}$$

Step 6. $$W(t) = \left[\frac{2m\, e^{m(\theta - t)}}{1 - e^{-2mT}}\right]\left[u(t) - u(t - T)\right] \tag{91}$$

Step 7. $$H(s) = \left[\frac{2m\, e^{m\theta}}{1 - e^{-2mT}}\right]\left[\frac{1 - e^{-mT} e^{-Ts}}{(s + m)}\right] \tag{92}$$

Step 8. One possible method of synthesizing the transfer function is shown in Figure 7. Operational amplifier elements are indicated by triangles. Actual values of the elements are not shown since they depend upon θ, T, the type of operational amplifier used and the anticipated signal swing. Either the delay element or its associated operational amplifier must provide a negative output with respect to the input in order that the final summing amplifier may perform the subtraction indicated in equation (92).

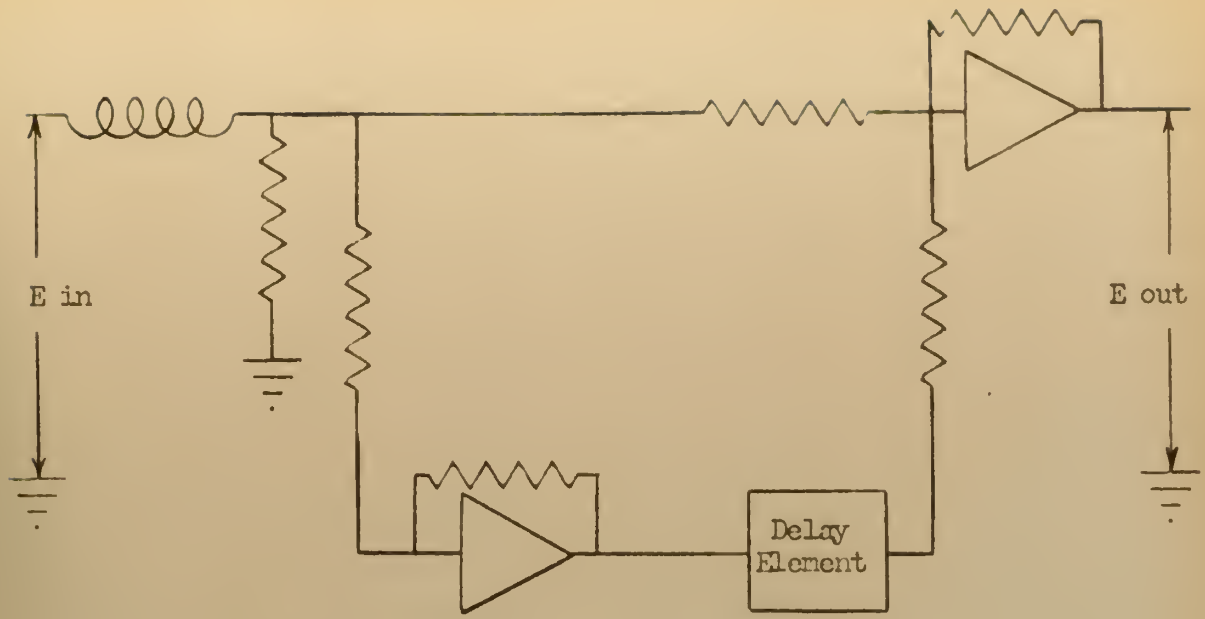


Figure 7. Exponential Predictor with Rectangular Weighting

The second case to be illustrated will be similar to the first with the exception that exponential weighting will be used.

Step 1. $g_1 = e^{mt}$ (93)

Step 2. $M(t) = e^{t/T}\,u(-t)$ (94)

Step 3. $B_{11} = \int_{-\infty}^{0} e^{2mt}\,e^{t/T}\,dt = \frac{1}{2m + 1/T}$ (95)

Step 4. $C_{11} = 2m + 1/T$ (96)

Step 5. $g_1(\theta) = e^{m\theta}$ (97)

Step 6. $W(t) = e^{m\theta}\left[2m + 1/T\right] e^{-(m + 1/T)\,t}\,u(t)$ (98)

Step 7. $Z(s) = \frac{e^{m\theta}\,(2m + 1/T)}{s + m + 1/T}$ (99)

Step 8. The synthesis of the filter as a driving point impedance is illustrated in Figure 8.

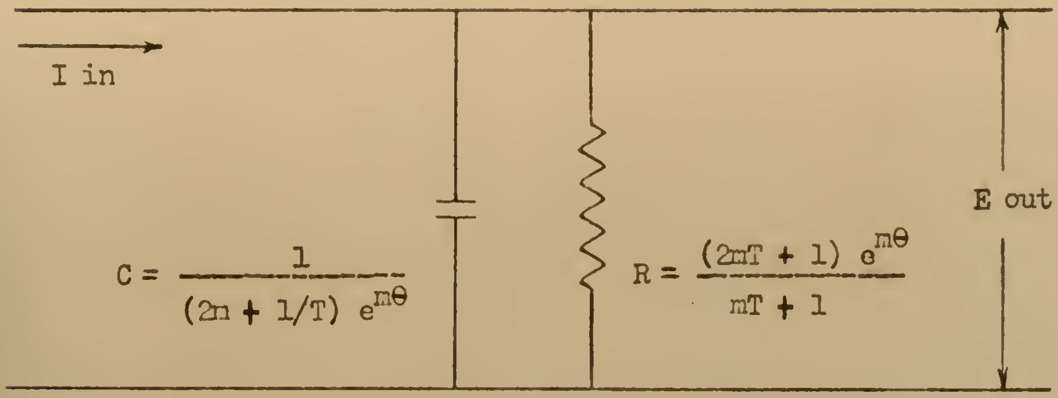


Figure 8. Exponential Predictor with Exponential Weighting

It may be noted that when m is negative, the values of T which will give a realizable network are restricted. This has physical significance in that it is necessary for the noise output to decay at a faster rate than the signal for a usable output.

4. Smoothing of Sinusoidal Functions.

The functions $a_1 \sin wt$, $a_2 \cos wt$, and the linear combination of the two are solutions of the second order linear differential equation with constant coefficients

$$f'' + C_1 f' + C_2 f = 0 \qquad \text{where } C_1 \text{ and } C_2 \text{ are constants} \qquad (100)$$

In the following example a noisy sinusoidal function with an angular frequency of one radian per second is to be smoothed. No prediction is desired and the error weighting is of the exponential type with a

smoothing time constant of one second.

Step 1. $$g_1 = \cos t \tag{101}$$

$$g_2 = \sin t \tag{102}$$

Step 2. $$M(t) = e^{t}\, u(-t) \tag{103}$$

Step 3. $$\begin{bmatrix} B_{jk} \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix} \tag{104}$$

Step 4. $$\begin{bmatrix} C_{ki} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \tag{105}$$

Step 5. $$g_1(0) = 1 \tag{106}$$

$$g_2(0) = 0 \tag{107}$$

Step 6. $$W(t) = (2 \cos t - \sin t)\, e^{-t}\, u(t) \tag{108}$$

Step 7. $$Z(s) = \frac{2s + 1}{s^2 + 2s + 2} \tag{109}$$

Step 8. Figure 9 shows the synthesis of this filter as a driving point impedance.

As might have been anticipated, the filter is a resonant circuit. At an angular frequency of one radian per second its impedance is purely resistive and equal to one ohm. The size of the elements should cause no alarm since they depend upon the desired frequency and impedance level of the filter. For very low frequencies, it might be convenient to use the mechanical analogues of the electrical elements.

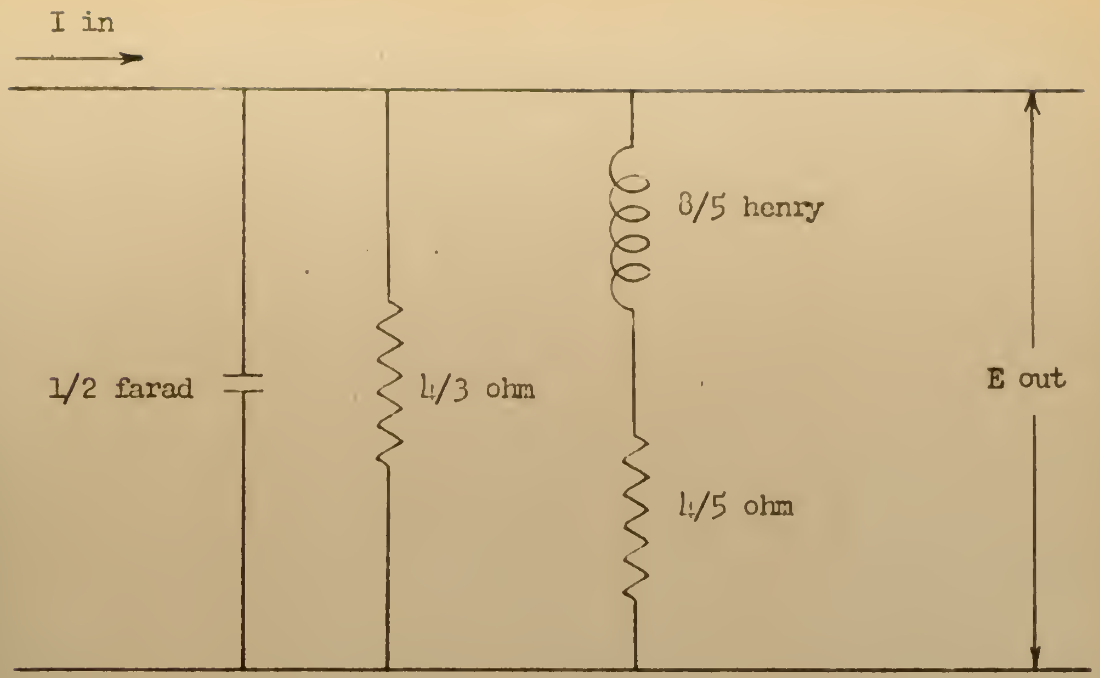


Figure 9. Filter for Smoothing Sinusoidal Functions

It appears possible to construct a filter for recurrent square or sawtooth waves by designing them for the fundamental frequency and a suitable number of the proper harmonics. A major difficulty with this method is that of finding the inverse of a matrix with a large number of elements.

5. Polynomial Filters.

As an example of the design of polynomial filters, the steps for constructing a filter which smooths and predicts a first order polynomial or ramp will be shown. The error weighting is of the exponential type with a smoothing time constant of one second.

Step 1. $g_1 = 1$ (110)

$g_2 = t$ (111)

Step 2. $M(t) = e^{t}\, u(-t)$ (112)

Step 3. $$\begin{bmatrix} B_{jk} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \tag{113}$$

Step 4. $$\begin{bmatrix} C_{ki} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \tag{114}$$

Step 5. $$g_1(\theta) = 1 \tag{115}$$

$$g_2(\theta) = \theta \tag{116}$$

Step 6. $$W(t) = \left[(2 + \theta) - t\,(1 + \theta)\right] e^{-t}\, u(t) \tag{117}$$

Step 7. $$Z(s) = \frac{(2 + \theta)\, s + 1}{(s + 1)^2} \tag{118}$$

Step 8. The synthesis of the filter as a driving point impedance is shown in Figure 10.

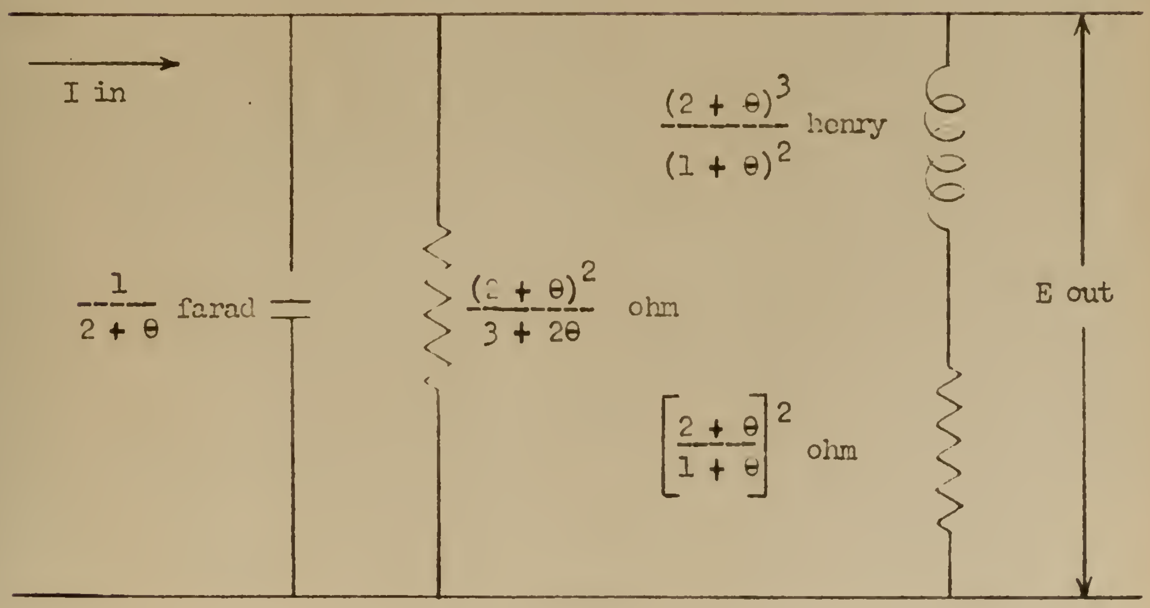

I in
$\frac{1}{2+\theta}$ farad
$\frac{(2+\theta)^2}{3+2\theta}$ ohm
$\frac{(2+\theta)^3}{(1+\theta)^2}$ henry
$\left[\frac{2+\theta}{1+\theta}\right]^2$ ohm
E out

Figure 10. Predictor for a First Order Polynomial

Tables 1 and 2 give the inverse matrices for polynomials up to and including the fifth order. Table 1 is for rectangular weighting with a smoothing time of 1 second while Table 2 is for exponential weighting with a smoothing time constant of 1 second. If T is other than 1 second, the tables may still be used by scaling the problem properly. For example, the expression for the weighting function of a predictor for a first order polynomial using rectangular weighting and a smoothing time of 1 second is

$$W(t) = \left[4 + 6\,\theta - (6 + 12\,\theta)\,t\right]\left[u(t) - u(t - 1)\right] \qquad (119)$$

and the corresponding expression for a general smoothing time T is

$$W(t, T) = \frac{1}{T}\left[4 + 6\,\frac{\theta}{T} - (6 + 12\,\frac{\theta}{T})\,\frac{t}{T}\right]\left[u(t) - u(t - T)\right] \qquad (120)$$

Figures 11 and 12 show the family of polynomial smoothing filters derived from Table 2. These filters are of the low pass type with an increasing response to higher frequencies as the order of the polynomial is increased. The complex frequency transfer functions of the n th order polynomial smoothing filters shown in Figures 11 and 12 have been found to take the form

$$H(s) = 1 - \frac{s^{n+1}}{(s + 1)^{n+1}} \qquad (121)$$

Figure 13 shows the amplitude and phase response versus frequency for first order polynomial predictors similar to the one in Figure 10. Figure 14 shows the same information about the polynomial filters in Figures 11 and 12. A positive increase in prediction time or polynomial order evidently tends to increase the noise output of the filter.

Table 1.

Inverse or $\left[Q_{ki} \right]$ Matrices for Polynomials of Order m.

$M(t) = u(-t) - u(-t-1)$

$$m = 0 \quad \begin{bmatrix} 1 \end{bmatrix}$$

$$m = 1 \quad \begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix}$$

$$m = 2 \quad \begin{bmatrix} 9 & 36 & 30 \\ 36 & 192 & 180 \\ 30 & 180 & 180 \end{bmatrix}$$

$$m = 3 \quad \begin{bmatrix} 16 & 120 & 240 & 140 \\ 120 & 1200 & 2700 & 1680 \\ 240 & 2700 & 6480 & 4200 \\ 140 & 1680 & 4200 & 2800 \end{bmatrix}$$

$$m = 4 \quad \begin{bmatrix} 25 & 300 & 1050 & 1400 & 630 \\ 300 & 4800 & 18900 & 26880 & 12600 \\ 1050 & 18900 & 79380 & 117600 & 56700 \\ 1400 & 26880 & 117600 & 179200 & 88200 \\ 630 & 12600 & 56700 & 88200 & 44100 \end{bmatrix}$$

$$m = 5 \quad \begin{bmatrix} 36 & 630 & 3360 & 7560 & 7560 & 2772 \\ 630 & 14700 & 88200 & 211680 & 220500 & 83160 \\ 3360 & 88200 & 564480 & 1411200 & 1512000 & 582120 \\ 7560 & 211680 & 1411200 & 3628800 & 3969000 & 1552320 \\ 7560 & 220500 & 1512000 & 3969000 & 4410000 & 1746360 \\ 2772 & 83160 & 582120 & 1552320 & 1746360 & 698544 \end{bmatrix}$$

Table 1.

Inverse or $\left[C_{ki}\right]$ Matrices for Polynomials of Order m.

$M(t) = u(-t) - u(-t-1)$

$$m = 0 \quad \begin{bmatrix} 1 \end{bmatrix} \qquad m = 1 \quad \begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix}$$

$$m = 2 \quad \begin{bmatrix} 9 & 36 & 30 \\ 36 & 192 & 180 \\ 30 & 180 & 180 \end{bmatrix} \qquad m = 3 \quad \begin{bmatrix} 16 & 120 & 240 & 140 \\ 120 & 1200 & 2700 & 1680 \\ 240 & 2700 & 6480 & 4200 \\ 140 & 1680 & 4200 & 2800 \end{bmatrix}$$

$$m = 4 \quad \begin{bmatrix} 25 & 300 & 1050 & 1400 & 630 \\ 300 & 4800 & 18900 & 26880 & 12600 \\ 1050 & 18900 & 79380 & 117600 & 56700 \\ 1400 & 26880 & 117600 & 179200 & 88200 \\ 630 & 12600 & 56700 & 88200 & 44100 \end{bmatrix}$$

$$m = 5 \quad \begin{bmatrix} 36 & 630 & 3360 & 7560 & 7560 & 2772 \\ 630 & 14700 & 88200 & 211680 & 220500 & 83160 \\ 3360 & 88200 & 564480 & 1411200 & 1512000 & 582120 \\ 7560 & 211680 & 1411200 & 3628800 & 3969000 & 1552320 \\ 7560 & 220500 & 1512000 & 3969000 & 4410000 & 1746360 \\ 2772 & 83160 & 582120 & 1552320 & 1746360 & 698544 \end{bmatrix}$$

Table 2.

Inverse or $\left[C_{li}\right]$ Matrices for Polynomials of Order m.

$M(t) == e^{t}\,u(-t)$

$$m = 0 \qquad \begin{bmatrix} 1 \end{bmatrix}$$

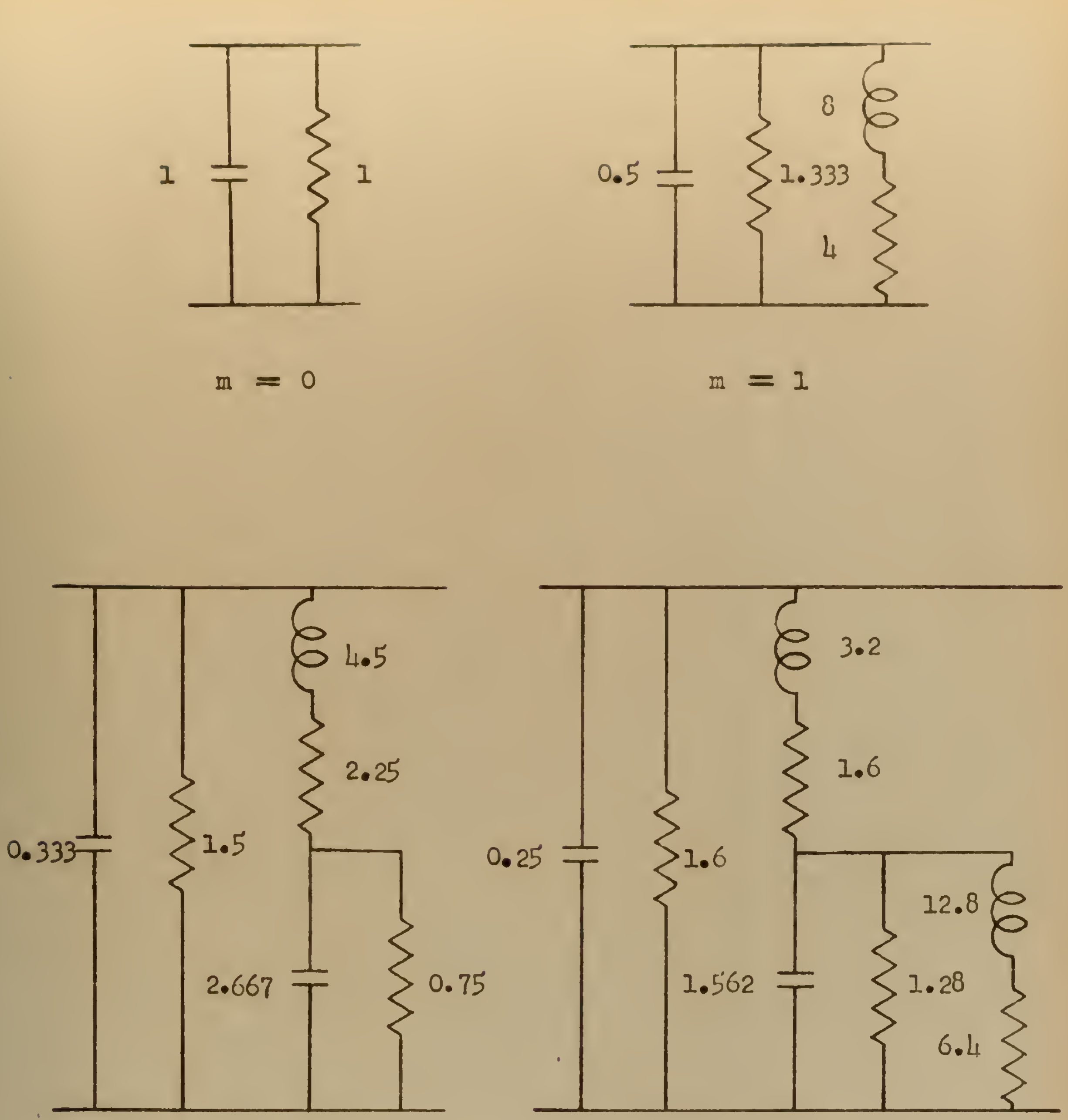
$$m = 1 \qquad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$m = 2 \qquad \begin{bmatrix} 3 & 3 & 1/2 \\ 3 & 5 & 1 \\ 1/2 & 1 & 1/4 \end{bmatrix}$$

$$m = 3 \qquad \begin{bmatrix} 4 & 6 & 2 & 1/6 \\ 6 & 14 & 11/2 & 1/2 \\ 2 & 11/2 & 5/2 & 1/4 \\ 1/6 & 1/2 & 1/4 & 1/36 \end{bmatrix}$$

$$m = 4 \qquad \begin{bmatrix} 5 & 10 & 5 & 5/6 & 1/24 \\ 10 & 30 & 35/2 & 19/6 & 1/6 \\ 5 & 35/2 & 23/2 & 9/4 & 1/8 \\ 5/6 & 19/6 & 9/4 & 17/36 & 1/36 \\ 1/24 & 1/6 & 1/8 & 1/36 & 1/576 \end{bmatrix}$$

$$m = 5 \qquad \begin{bmatrix} 6 & 15 & 10 & 15/6 & 1/4 & 1/120 \\ 15 & 55 & 85/2 & 23/2 & 29/24 & 1/24 \\ 10 & 85/2 & 73/2 & 127/12 & 7/6 & 1/24 \\ 15/6 & 23/2 & 127/12 & 13/4 & 3/8 & 1/72 \\ 1/4 & 29/24 & 7/6 & 3/8 & 13/288 & 1/576 \\ 1/120 & 1/24 & 1/24 & 1/72 & 1/576 & 1/14400 \end{bmatrix}$$



m = 0 m = 1

m = 2 m = 3

Figure 11. Zero Through Third Order Polynomial Smoothing Filters.

$M(t) = e^t u(-t)$

All resistances, inductances, and capacitances are given in ohms, henries, and farads respectively.

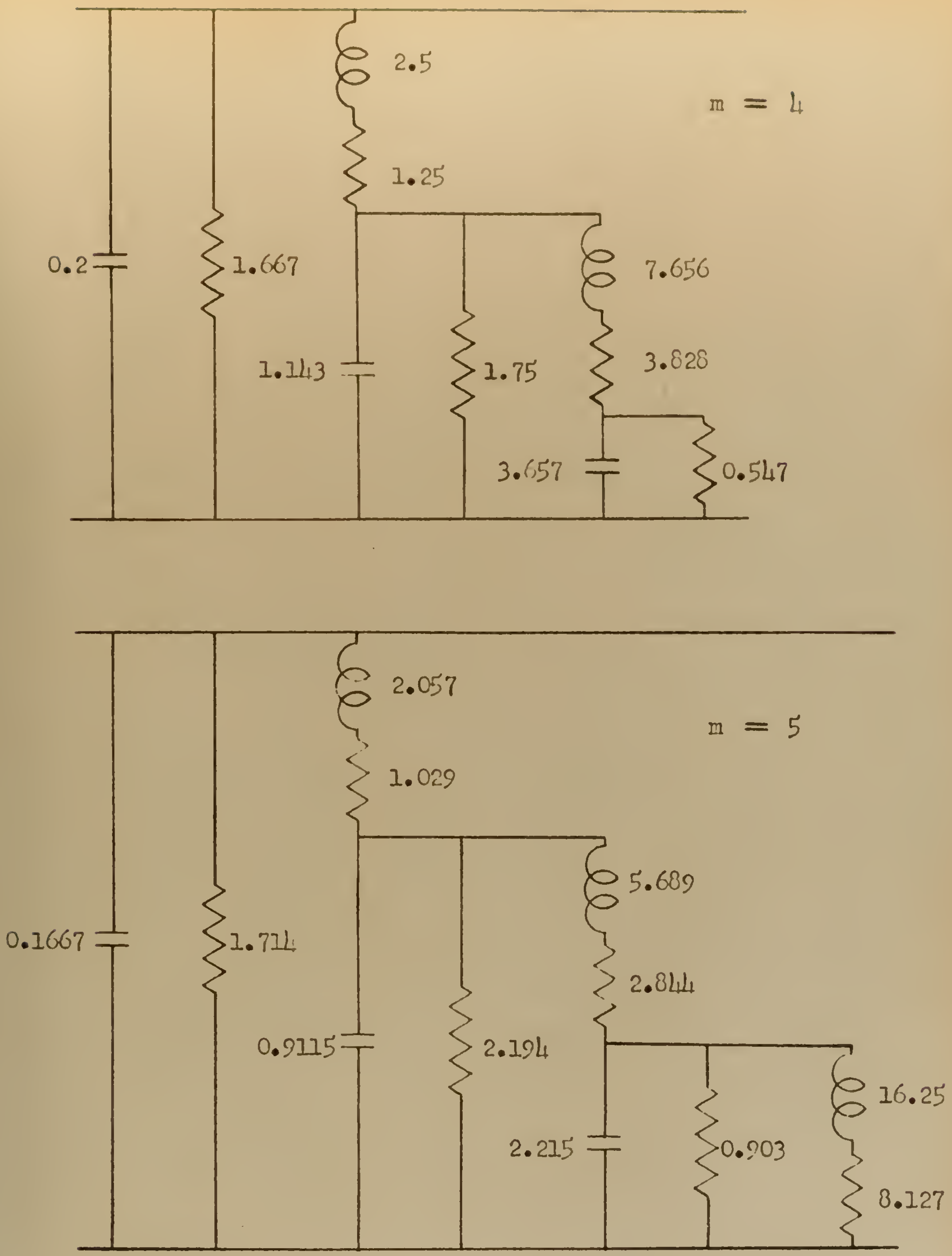


Figure 12. Fourth and Fifth Order Polynomial Smoothing Filters.

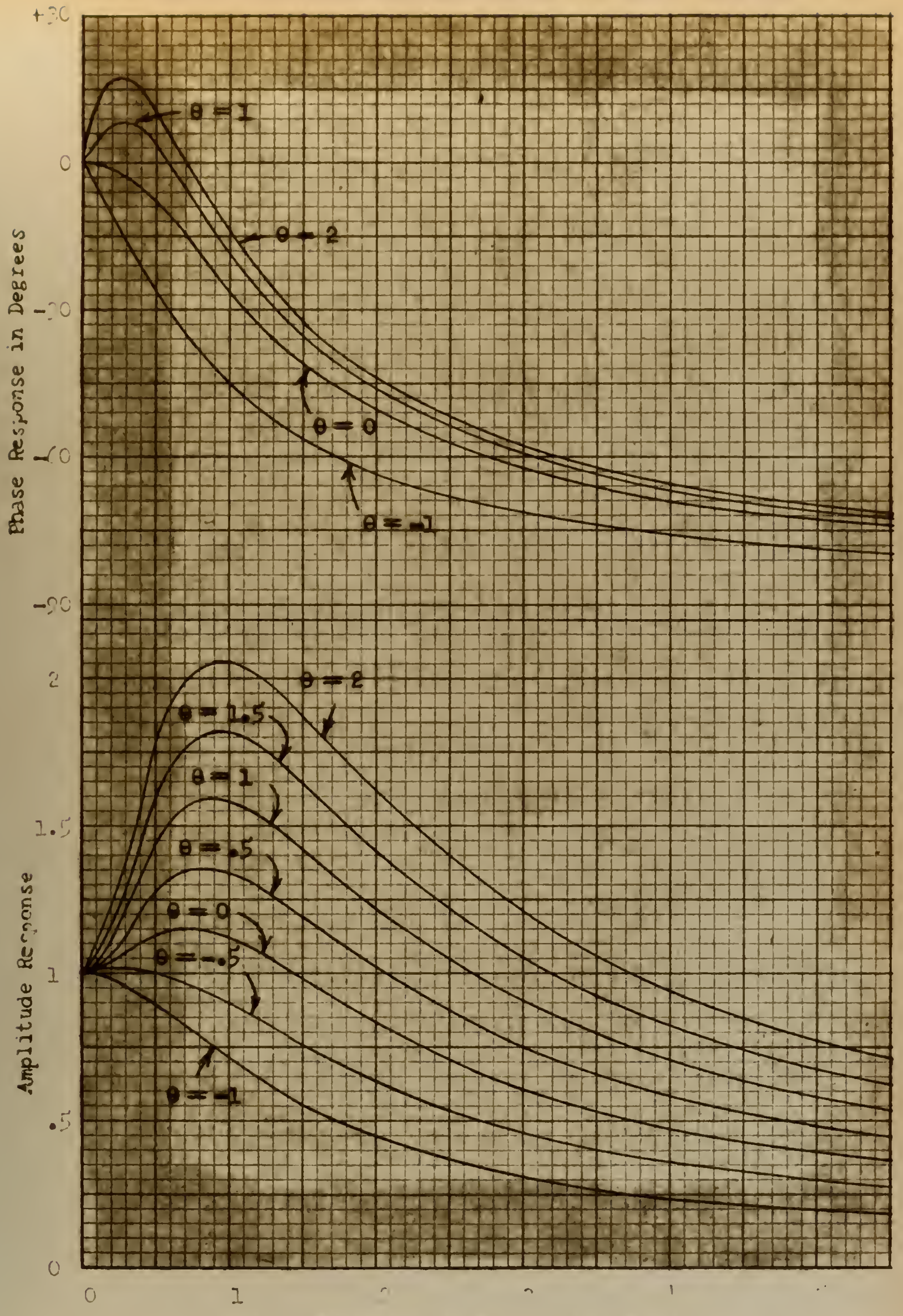
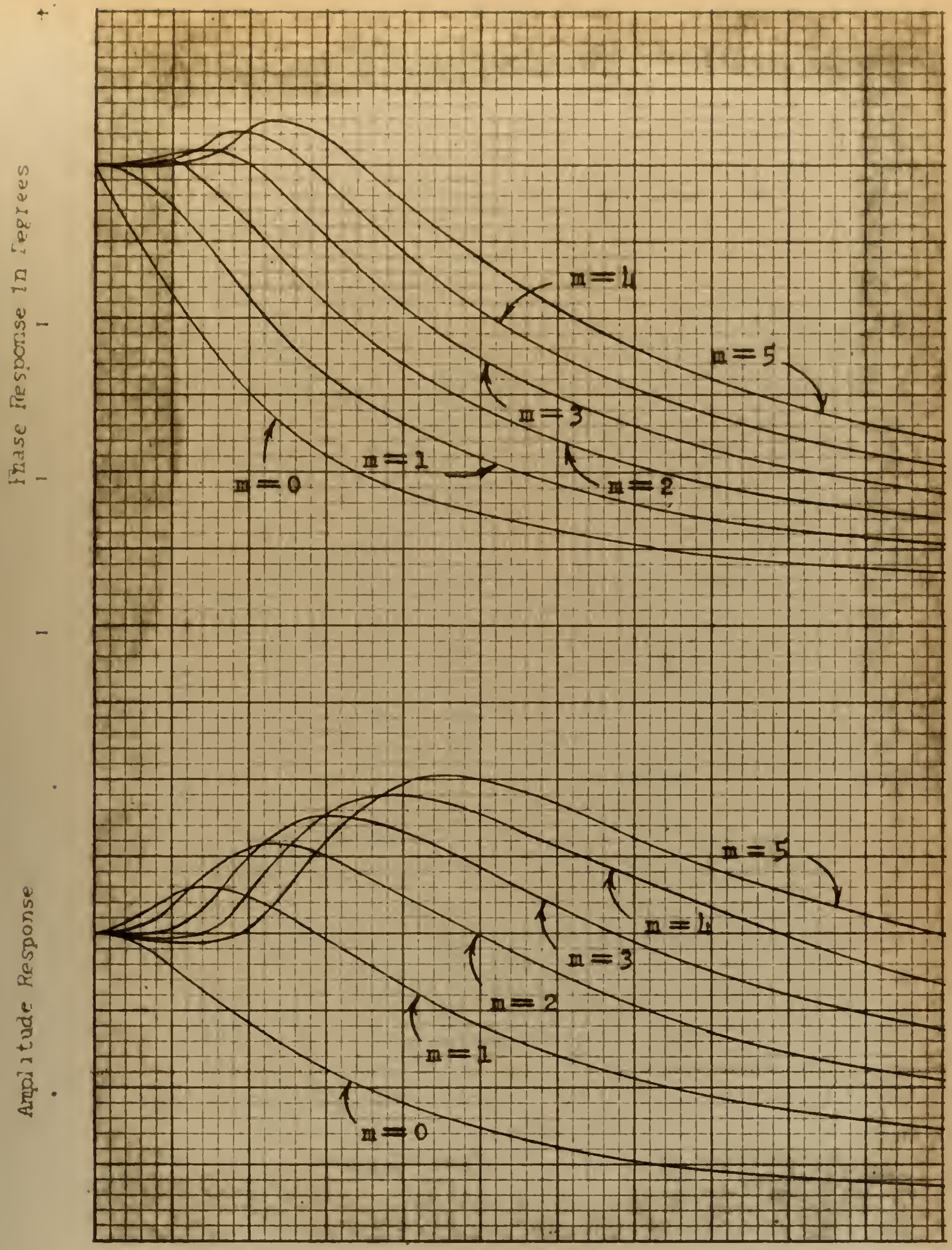


Figure [illegible]. Amplitude and Phase [illegible]


Phase Response in Degrees
m=4
m=5
m=3
m=1
m=2
m=0
Amplitude Response
m=5
m=4
m=3
m=2
m=1
m=0

6. Filters for a Combination of Functions.

Since any additive or multiplicative combination of sinusoids, exponentials, and polynomials is a solution to a reduced equation of some linear differential equation with constant coefficients, it may be handled. Such functions as t sin t, e^t sin t, and t e^t sin t may thus be treated after the complete set of functions forming a closed set upon translation in time is found. A filter which is the additive combination of a first order polynomial and a sinusoid can be designed as follows.

Step 1. $$g_1 = \cos t \tag{122}$$

$$g_2 = \sin t \tag{123}$$

$$g_3 = 1 \tag{124}$$

$$g_4 = t \tag{125}$$

Step 2. $$M(t) = e^t u(-t) \tag{126}$$

Step 3.
$$\left[B_{jk}\right] = \begin{bmatrix} 3/5 & -1/5 & 1/2 & 0 \\ -1/5 & 2/5 & -1/2 & 1/2 \\ 1/2 & -1/2 & 1 & -1 \\ 0 & 1/2 & -1 & 2 \end{bmatrix} \tag{127}$$

Step 4.
$$\left[C_{ki}\right] = \begin{bmatrix} 12 & -4 & -14 & -6 \\ -4 & 8 & 8 & 2 \\ -14 & 8 & 20 & 8 \\ -6 & 2 & 8 & 4 \end{bmatrix} \tag{128}$$

Step 5. $g_1(0) = 1$ (129)

$g_2(0) = 0$ (130)

$g_3(0) = 1$ (131)

$g_4(0) = 0$ (132)

Step 6. $W(t) = \left[6 - 2t - 2\cos t - 4\sin t\right] e^{-t} u(t)$ (133)

Step 7. $Z(s) = \dfrac{4s^3 + 6s^2 + 6s + 2}{s^4 + 4s^3 + 7s^2 + 6s + 2}$ (134)

Step 8. Figure 15 illustrates one method of realizing this filter as a driving point impedance.

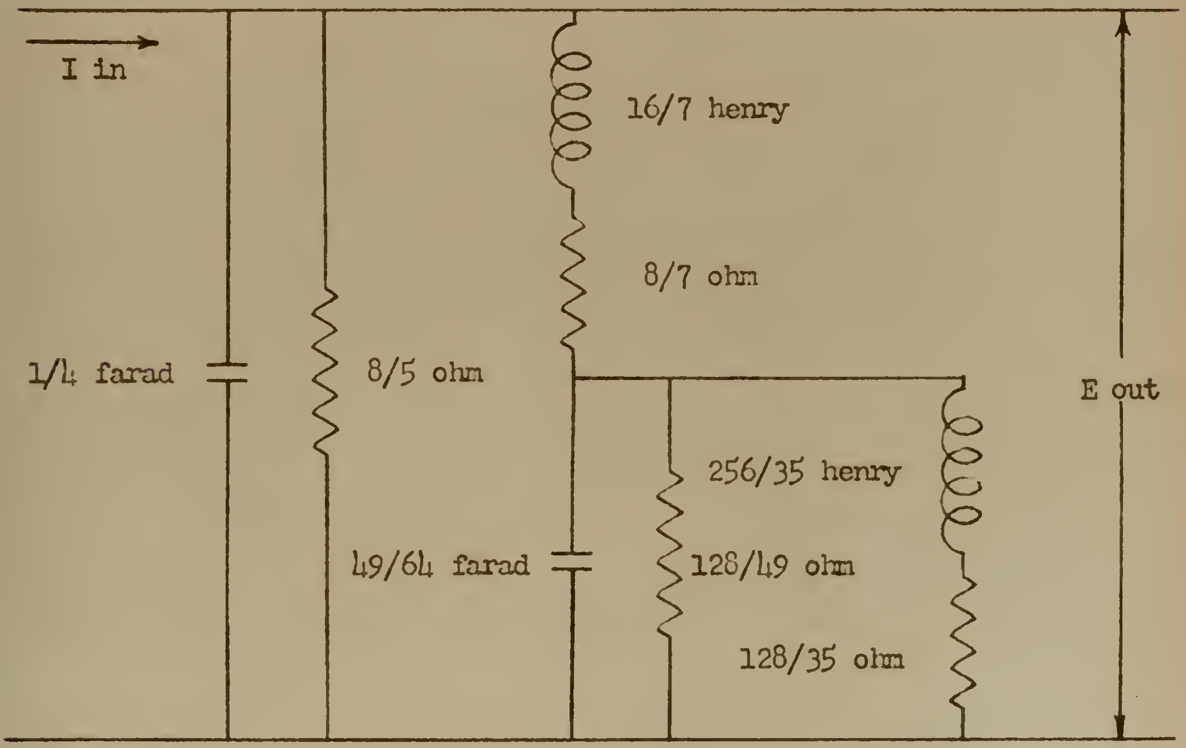


Figure 15. Smoothing Filter for a Ramp and Sine Wave.

From equation (134)

$$Z(j1) = 1 \tag{135}$$

$$Z(0) = 1 \tag{136}$$

Thus in the steady state the driving point impedance for a sinusoidal input with an angular frequency of one radian per second or for a constant input is a pure resistance of one ohm. When the input is equal to t, the output may be found by multiplying $Z(s)$ by the Laplace transform of t, that is $1/s^2$. The inverse Laplace transform of the product has a steady state term equal to t. The steady state output divided by the input is then t/t or 1. Although the filter is constructed with reactive elements, its steady state impedance to a sinusoid, a constant, and a ramp input appears to be a pure resistance of one ohm. A phenomenon akin to resonance is thus observed for functions of time other than sine waves.

7. Response of Various Polynomial Filters in the Absence of Noise.

In order to evaluate the polynomial filters, the response of the filters to standard input signals in the absence of noise can be calculated from equation (5). The standard signals chosen for this evaluation are

$$R_1 = 1 \qquad \text{(a constant)} \tag{137}$$

$$R_2 = 1 + t \qquad \text{(a ramp)} \tag{138}$$

$$R_3 = 1 + t + t^2 \qquad \text{(a quadratic)} \tag{139}$$

The graphs on the following pages show the desired response to these signals as dashed lines and the actual response as solid lines. For smoothing filters the desired steady state responses to R_1, R_2, and R_3 are 1, $1 + t$, and $1 + t + t^2$; for integrating filters, t, $t + t^2/2$, and $t + t^2/2 + t^3/3$; and for differentiators, 0, 1, and $1 + 2t$ respectively.

Figures 16 through 21 inclusive illustrate the response of smoothing filters designed for constants, ramps, and quadratics with both rectangular and exponential weighting. It should be noted that the steady state response of the rectangular filter is reached abruptly at the end of the smoothing period while the exponential filter requires five to ten time constants. Filters designed to smooth an m th order polynomial will also smooth lower order polynomials. If a polynomial of order $m + q + 1$ is put into the filter, the output will have an error which is a polynomial in t of order q, where q is a positive constant.

The response of polynomial smoothing filters to a unit step input is illustrated in Figures 22 and 23. As the order of the polynomial for which the filter is designed increases, the overshoot and frequency of transient oscillation is observed to increase.

Figures 24, 25, and 26 show the response of filters designed to predict a ramp function with four different values of prediction time and three different error memory functions. The influence of the error memory function upon the transient response can be clearly seen.

Responses of filters derived by dividing the complex frequency transfer functions, $H(s)$, for the constant and ramp smoothing filters by the variable s are shown in Figures 27 through 30. These curves indicate that a filter which is to integrate polynomials up to the m th order should be

based upon polynomial smoothing filters of the m + p th order, where p is the number of integrations to be performed. This might be anticipated from the fact that an integrator increases the order of the output polynomial as compared with the input polynomial.

Differentiators have responses as illustrated by Figures 31 through 36. These filters differentiate all polynomials up to the order m if they are based on polynomial smoothing filters of at least the order m - q, where q is the number of differentiations. If the rectangular or exponential error memory functions are used, the response curves show discontinuities in the output. These discontinuities indicate that the filters have an infinite frequency response and hence would not be very practical for eliminating high frequency noise components. The discontinuities result from the fact that the derivative of a unit step function is the unit impulse or Dirac delta function and may be eliminated by choosing error memory functions which never have an infinite slope. The responses of two filters having such error memory functions are shown in Figures 35 and 36.

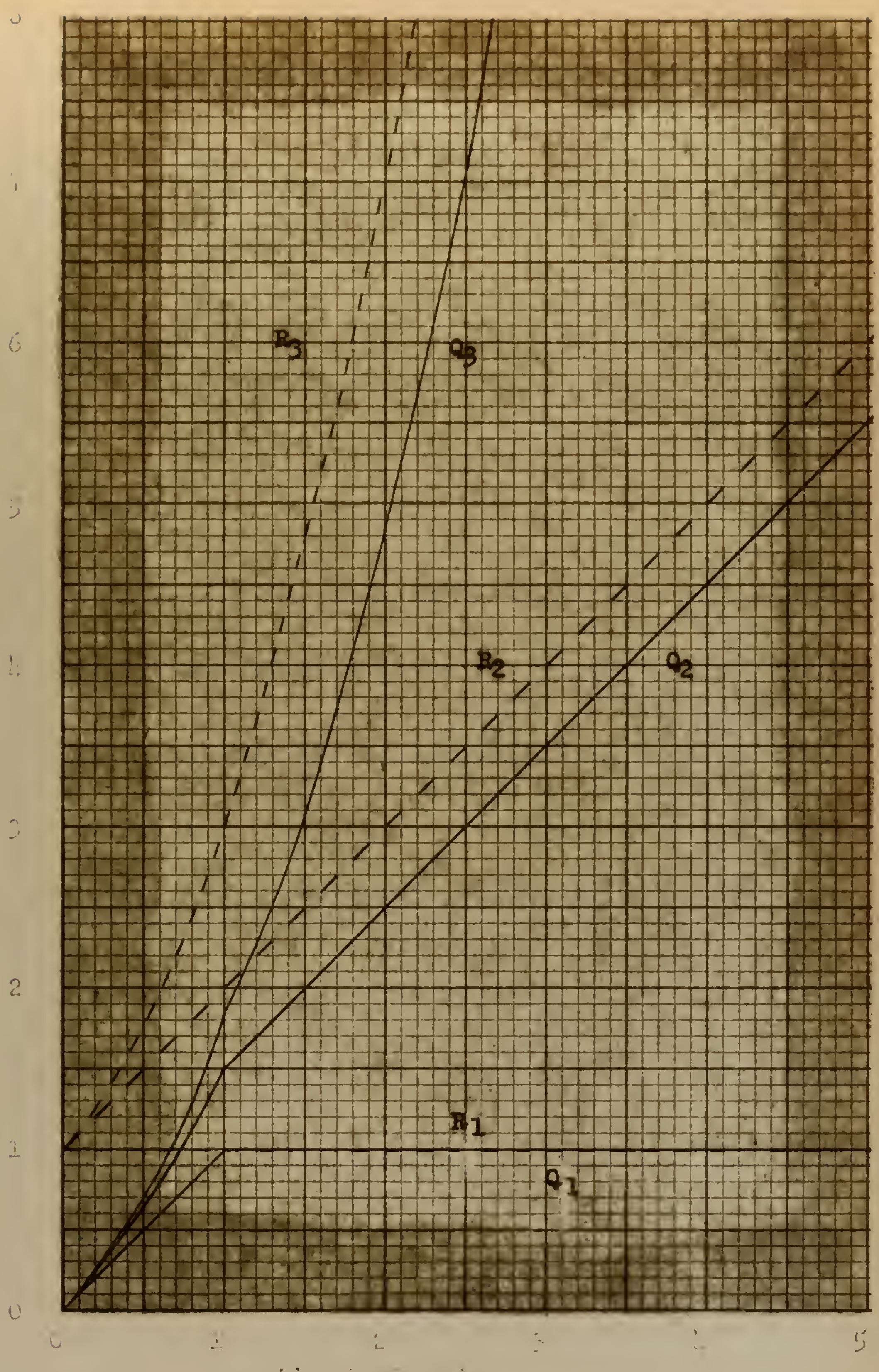


Figure 16. Response of Rectangularly Weighted [illegible] Filter for a Constant.

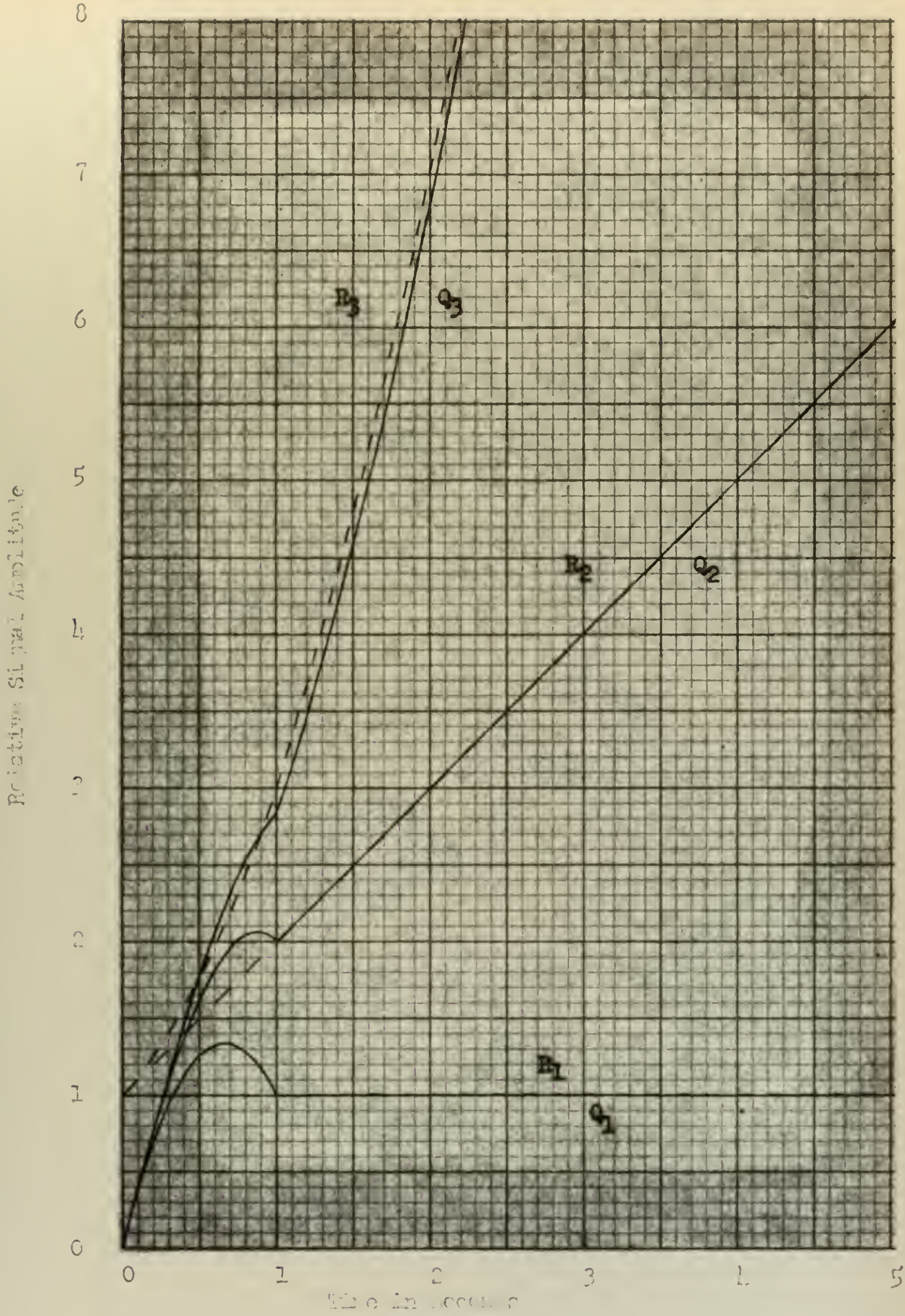


Figure 17. Response of Rectangularly Weighted Smoothing Filter for a Ramp.

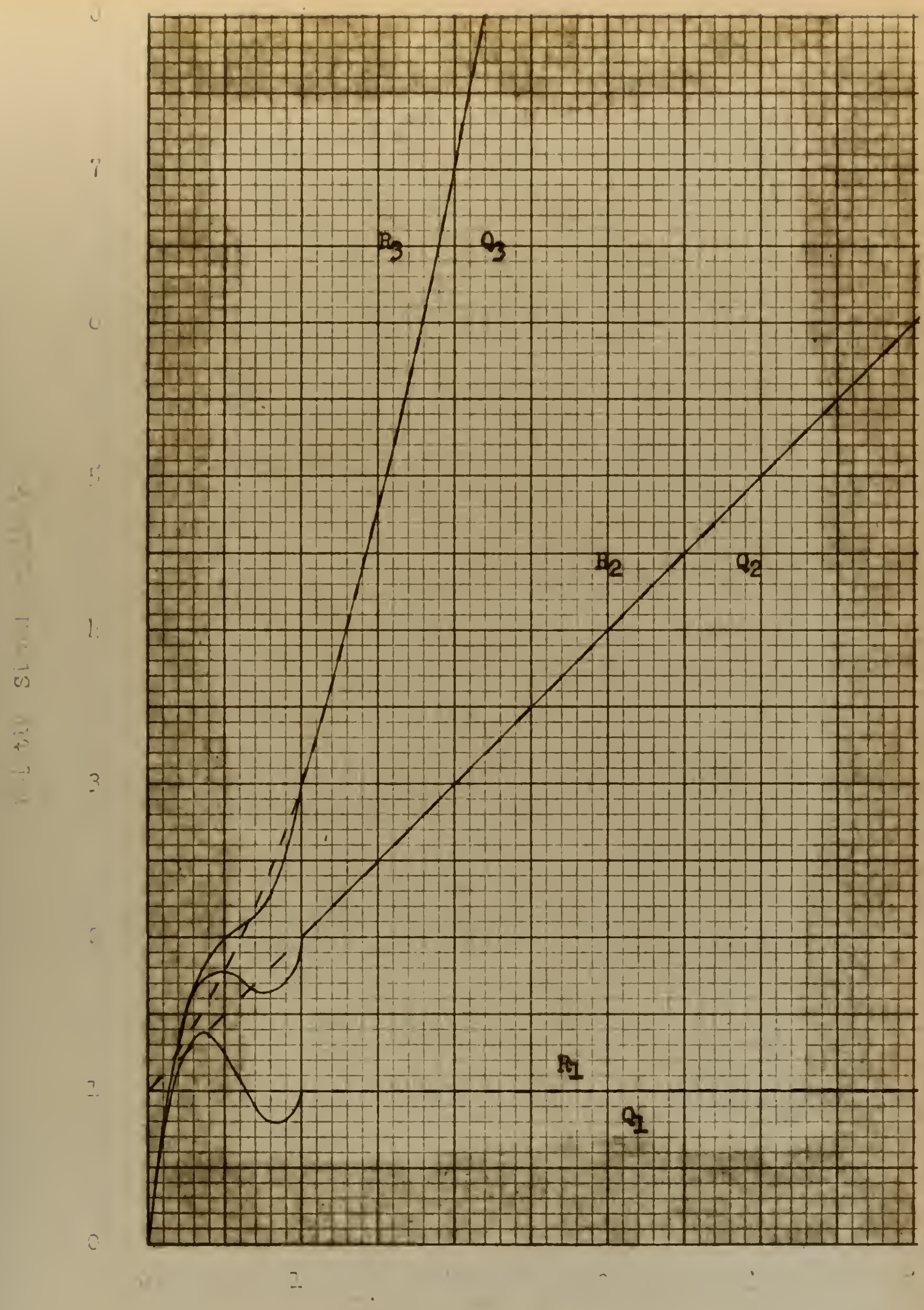


Figure [illegible]

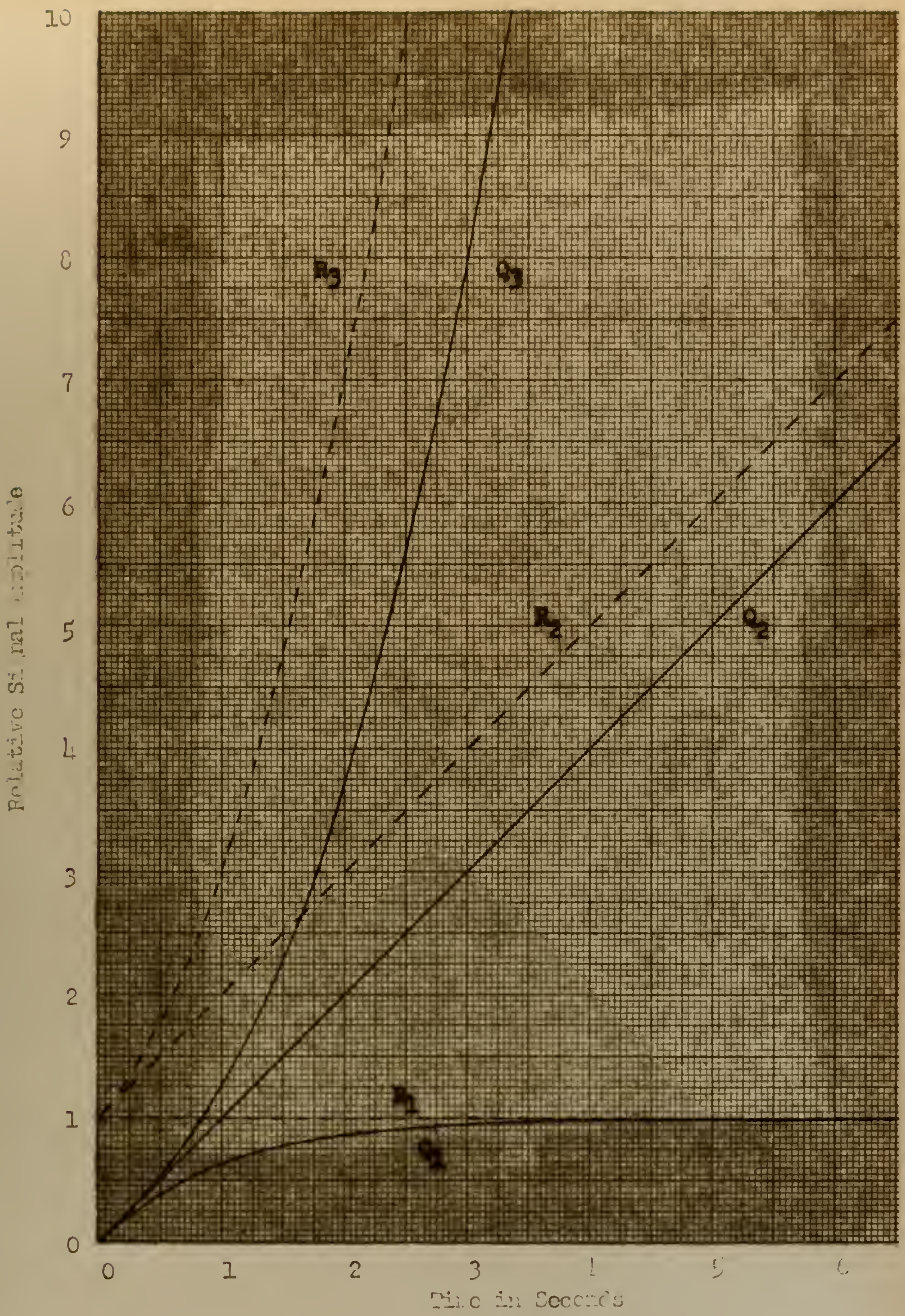


Figure 19. Response of Exponentially Weighted Predicting Filter for a Constant.

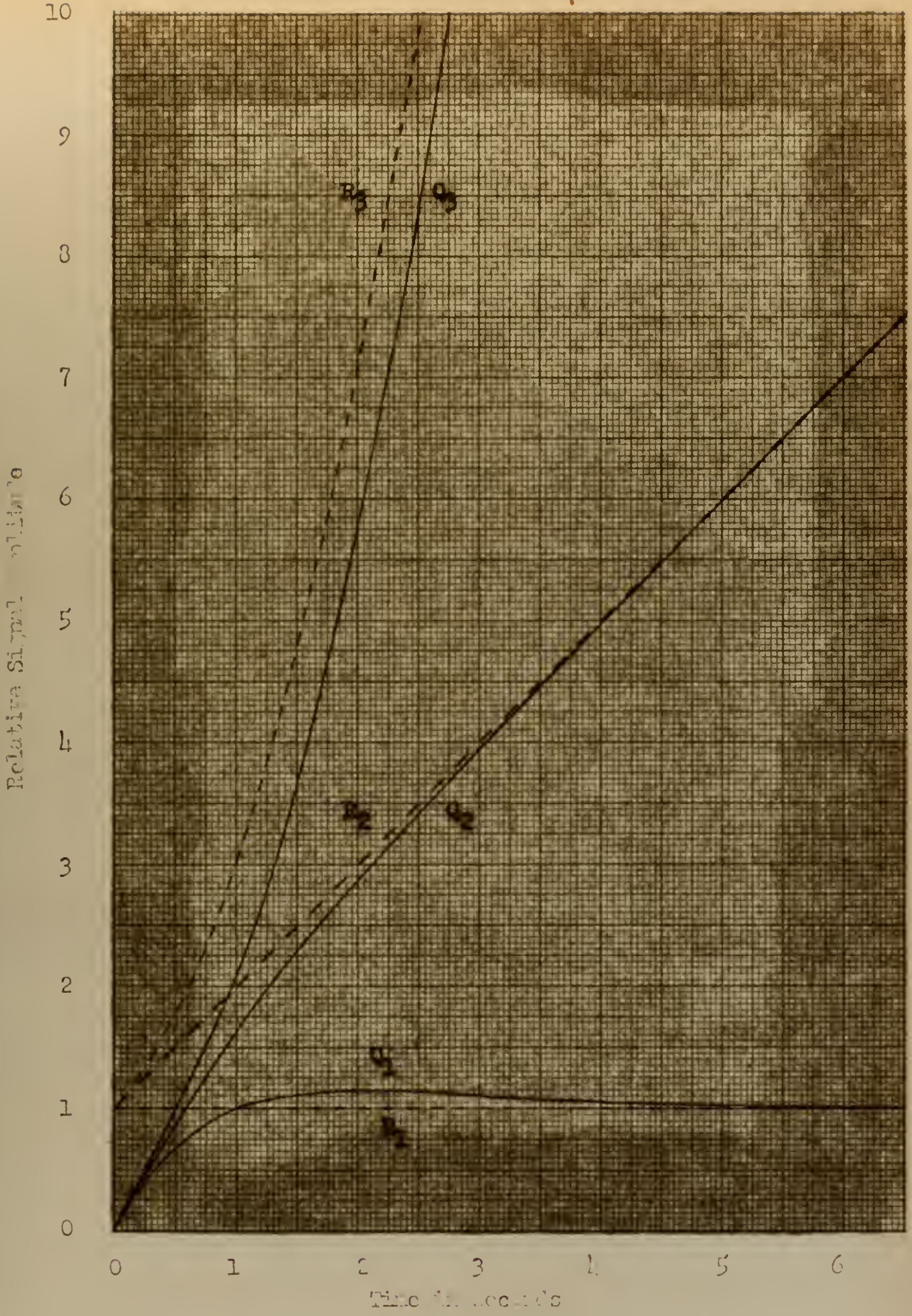


Figure 20. Response of Exponentially Weighted Smoothing Filter for a Ramp.

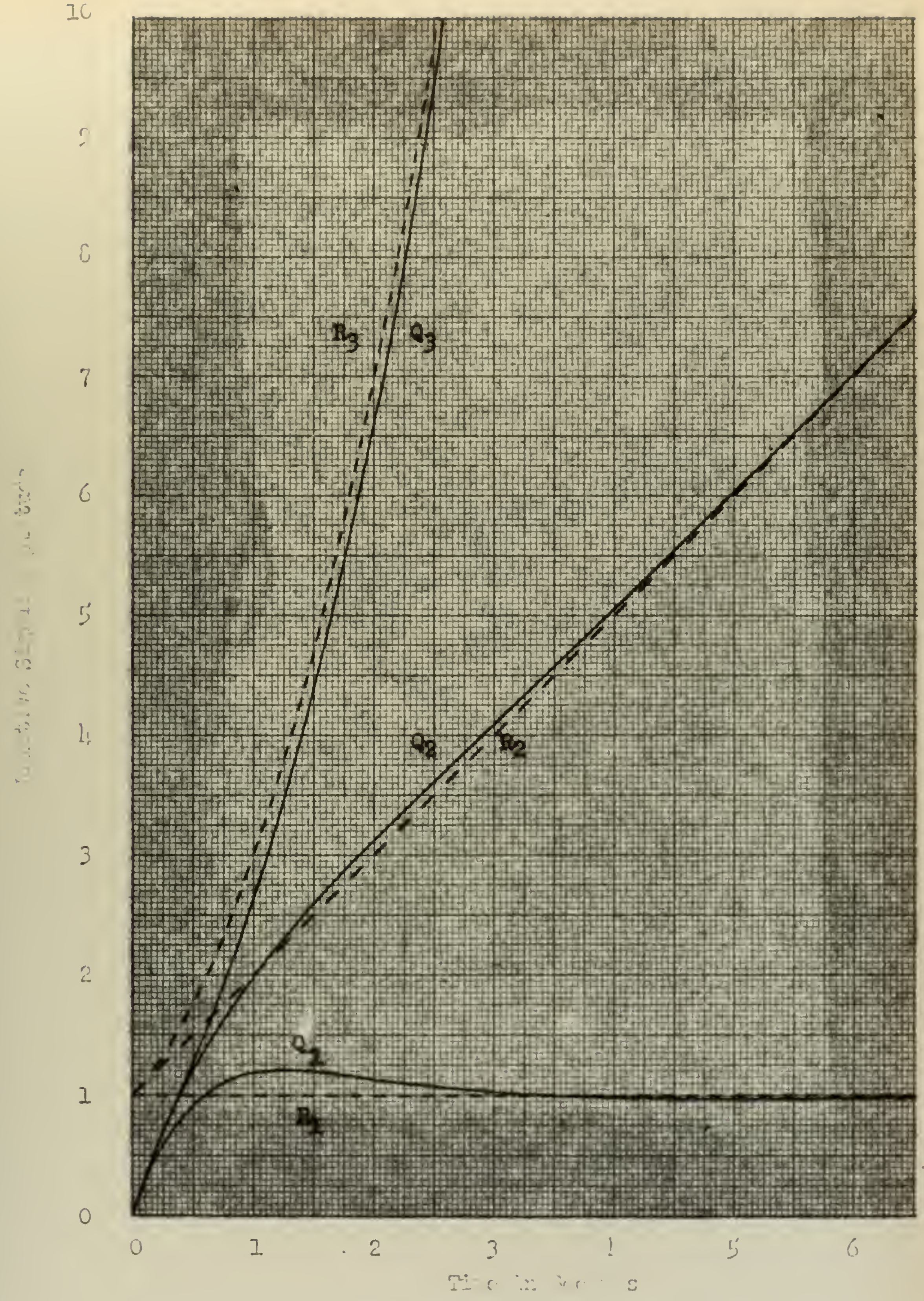


Figure 11. Response of theoretically [illegible] to [illegible] filter for a quadratic.

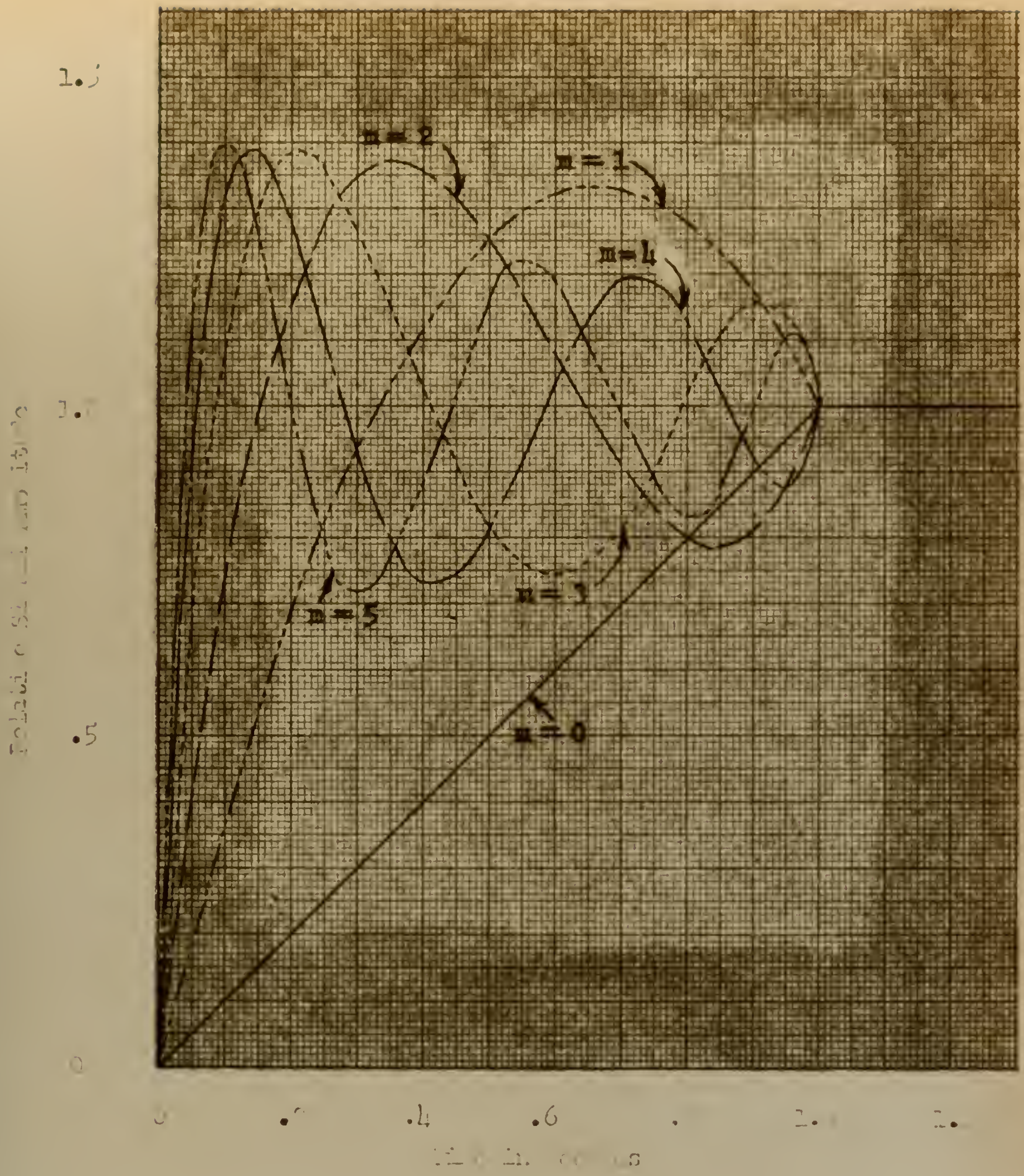


Figure [illegible]. [illegible] to step input.

Order of polynomial for which filter is designed = m

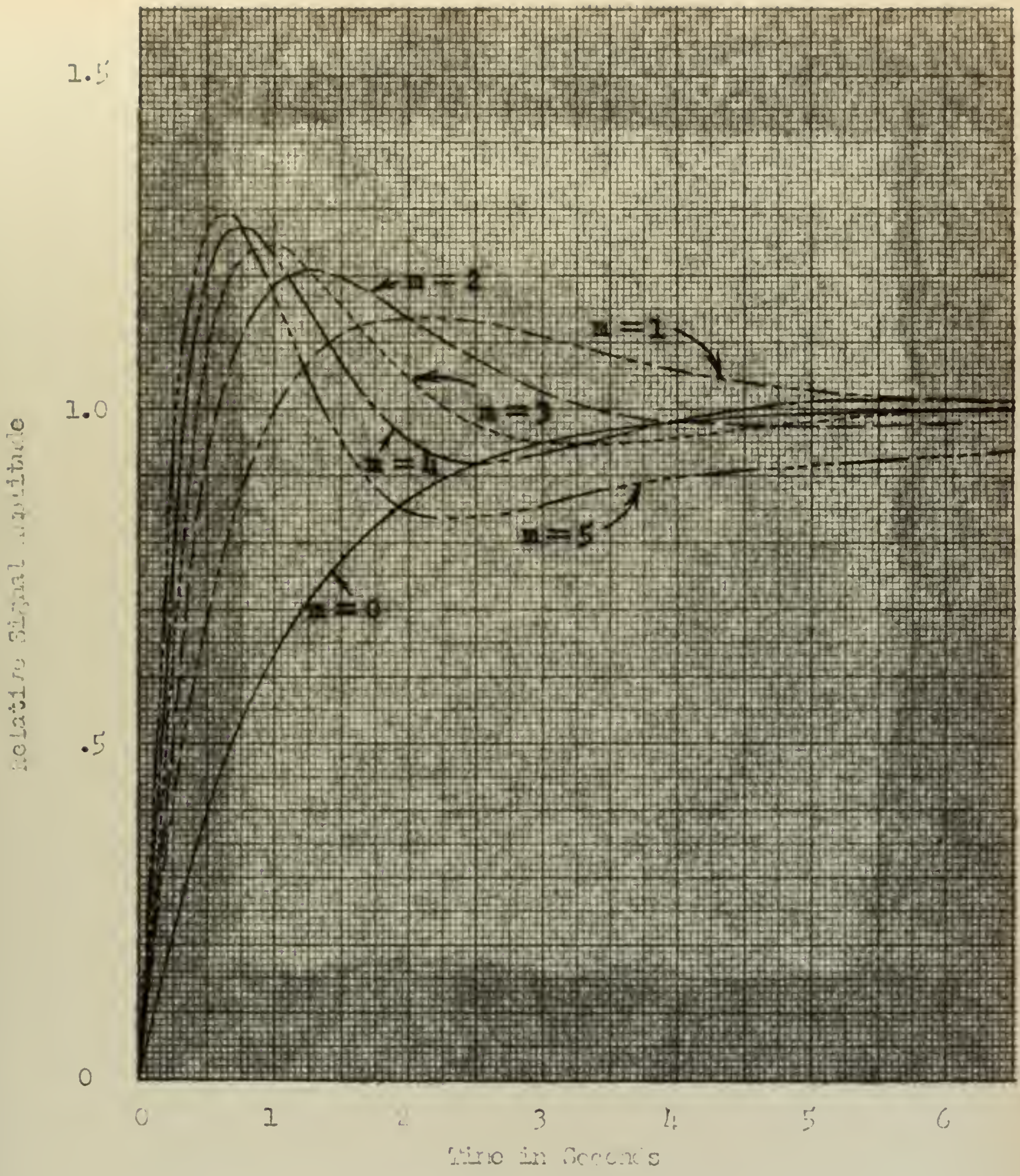


Figure 23. Response of Exponentially Weighted Polynomial Smoothing Filters to Step Input.

Order of polynomial for which filter is designed = m.

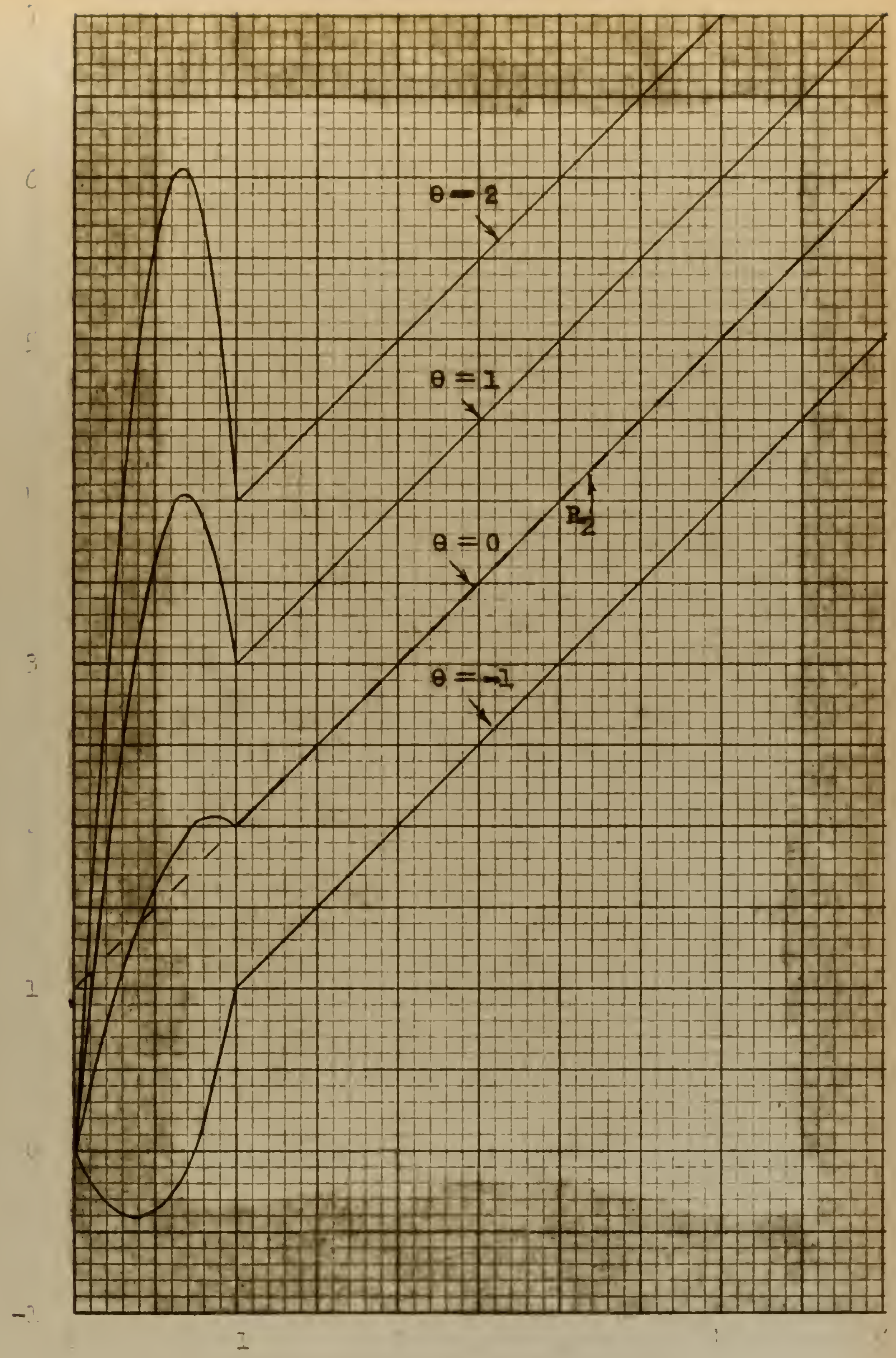
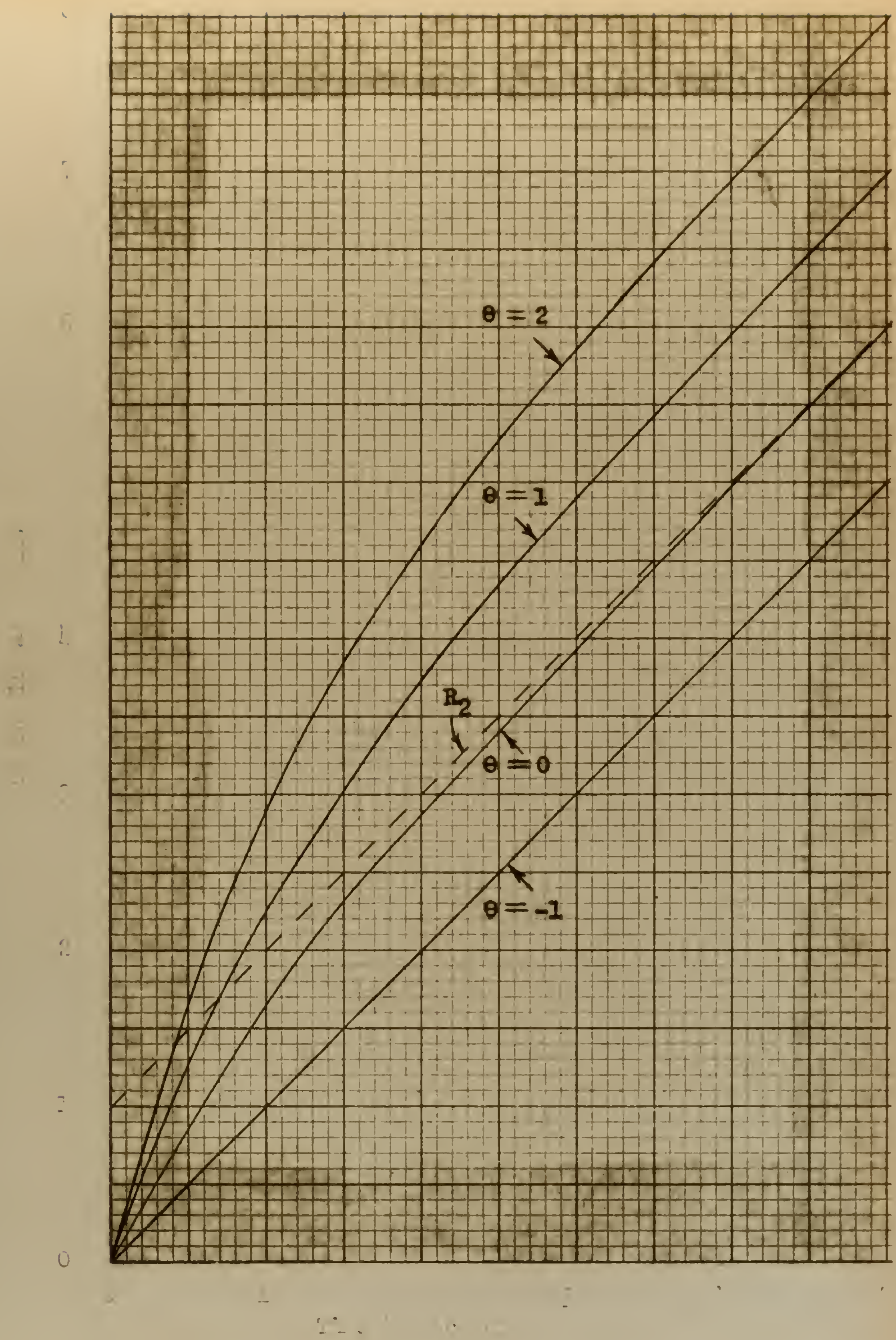


Figure 7b. [illegible]



θ = 2
θ = 1
R_2
θ = 0
θ = -1
0

[illegible]

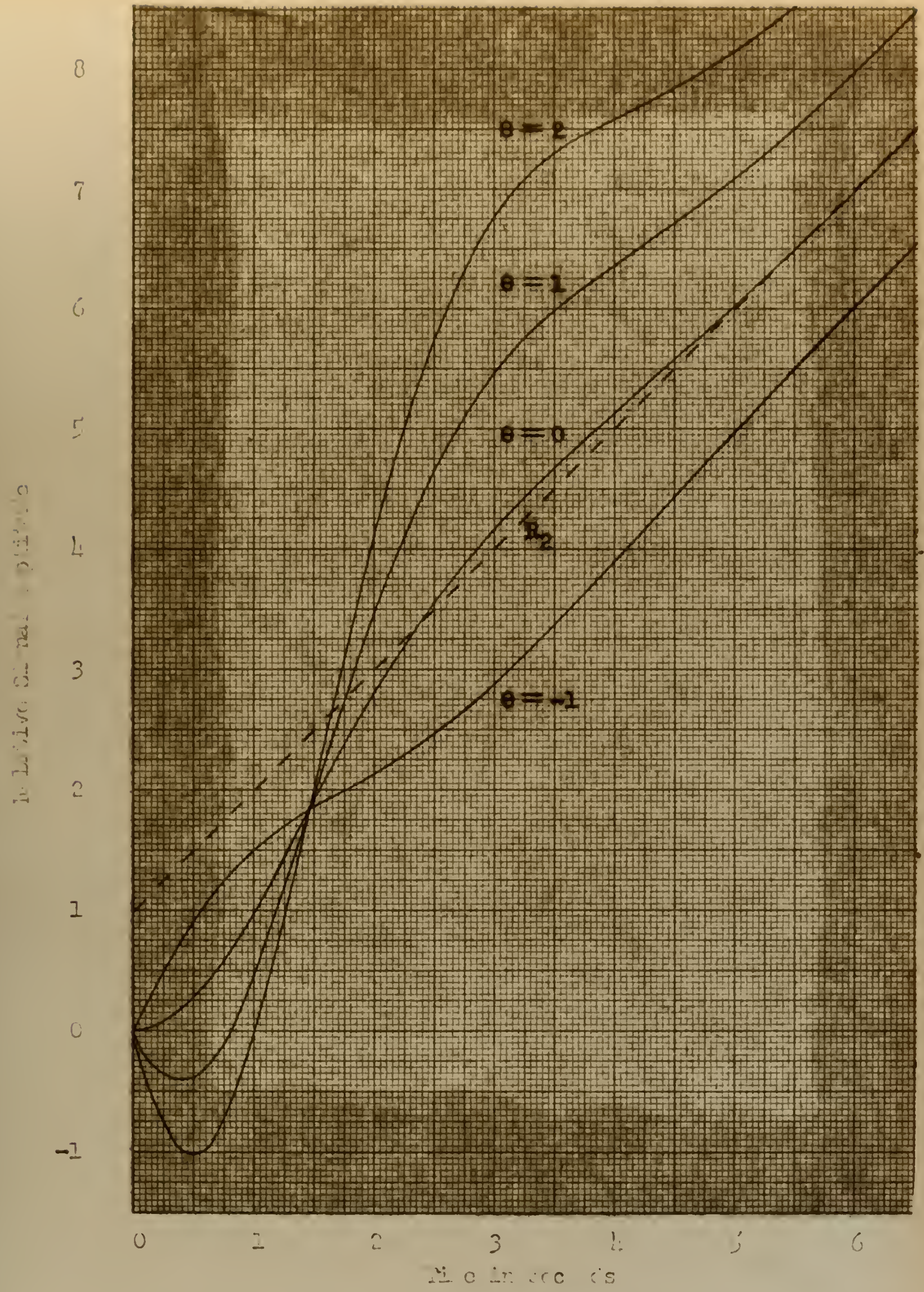


Figure 26. Response of [illegible] Statistically Weighted Predictor for Ramp.

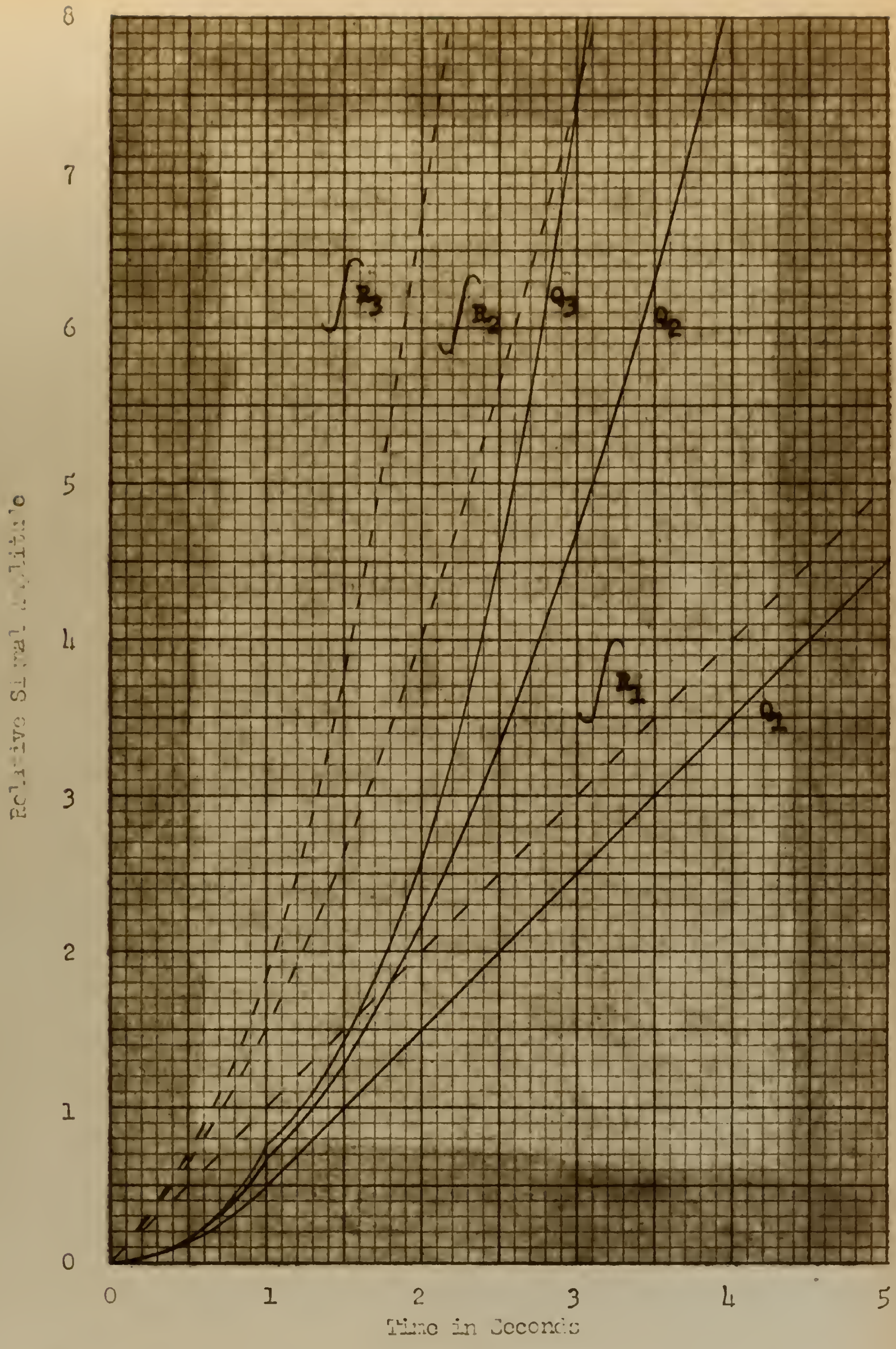


Figure 17. Response of Integrator Derived from Rectangularly Weighted Smoothing Filter for a Constant.

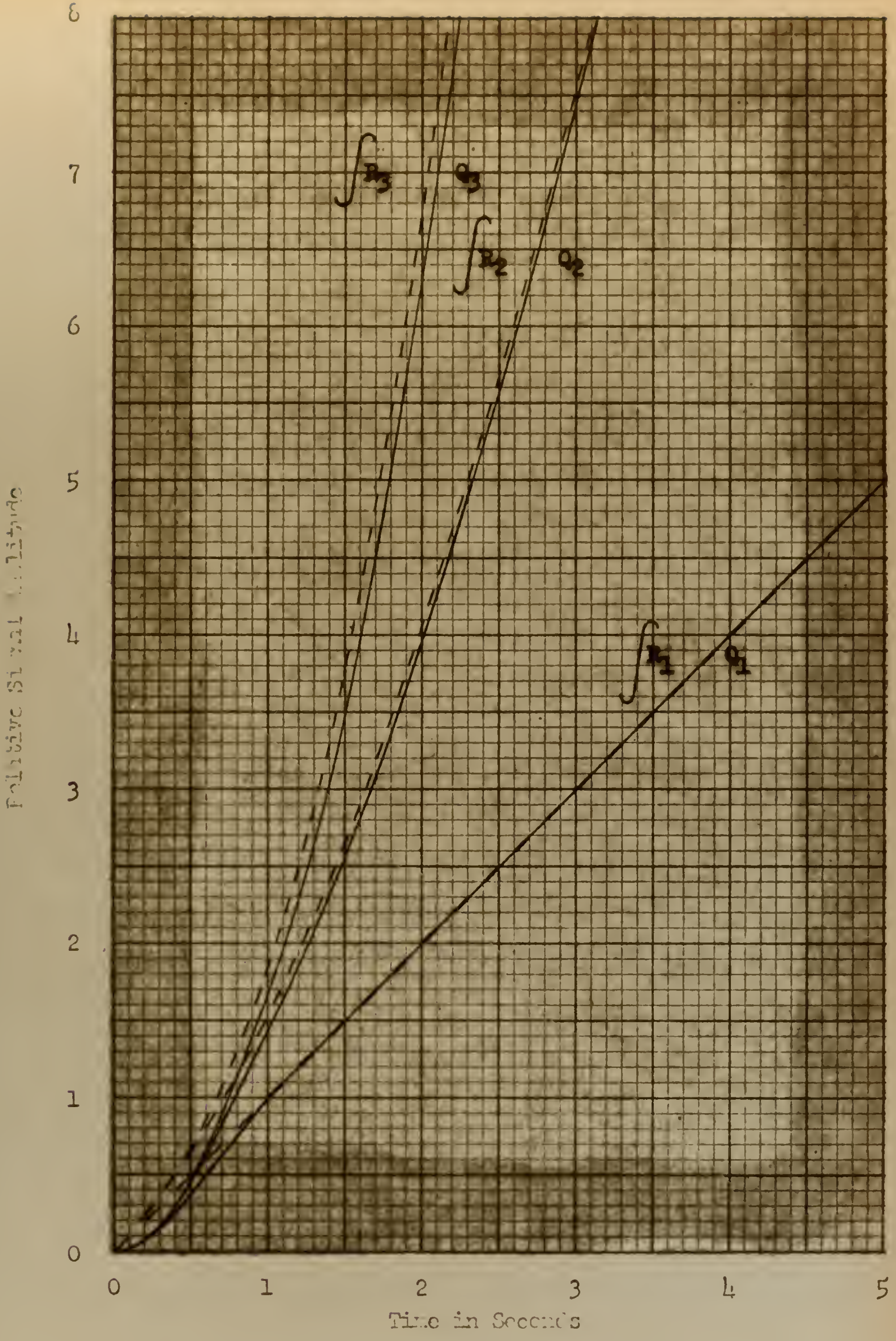
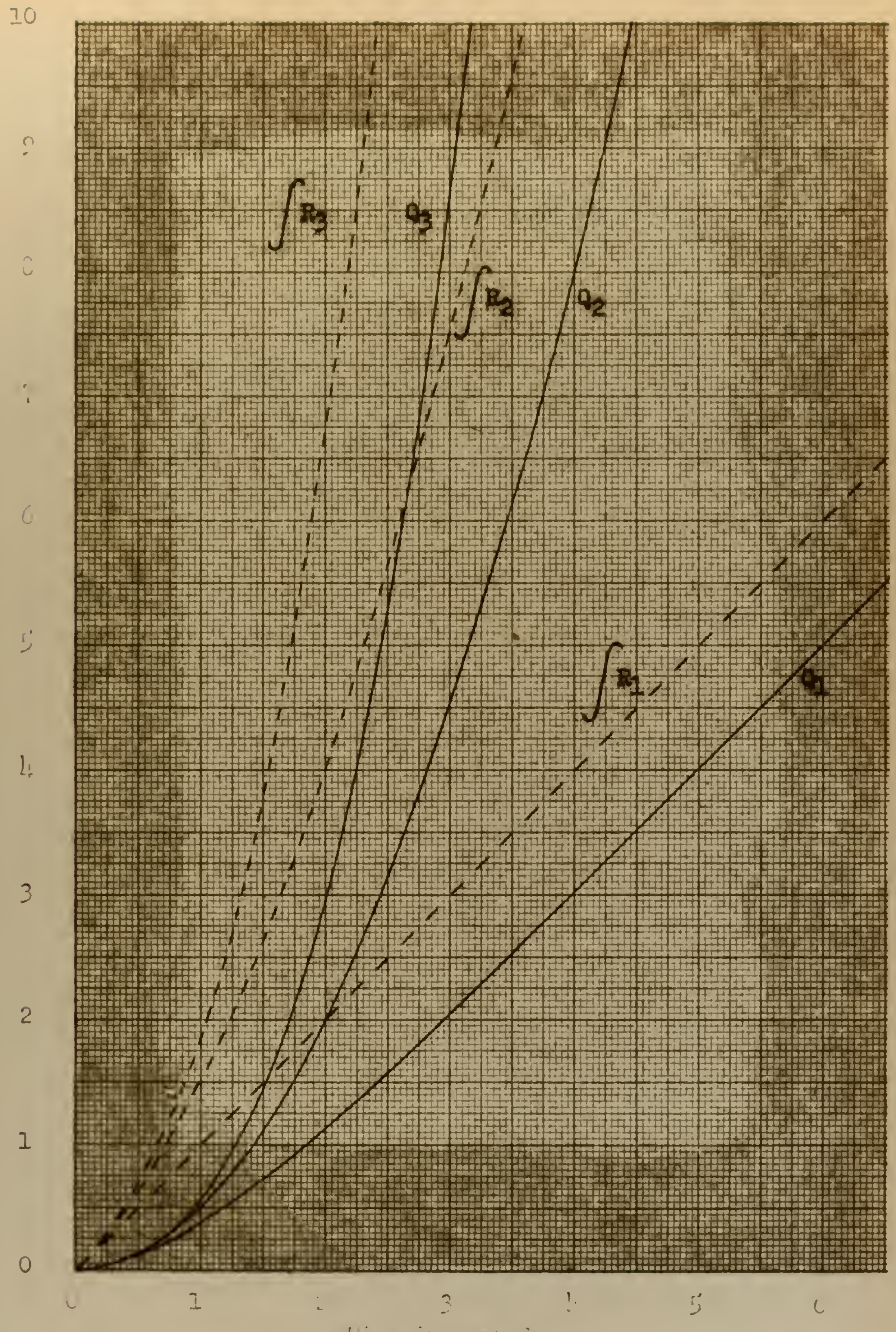


Figure 28. Response of Integrator Derived from Rectangularly Weighted Smoothing Filter for a Ramp.


10
9
8
7
6
5
4
3
2
1
0
0
1
2
3
4
5
6
∫R3
Q3
∫R2
Q2
∫R1
Q1

[illegible]

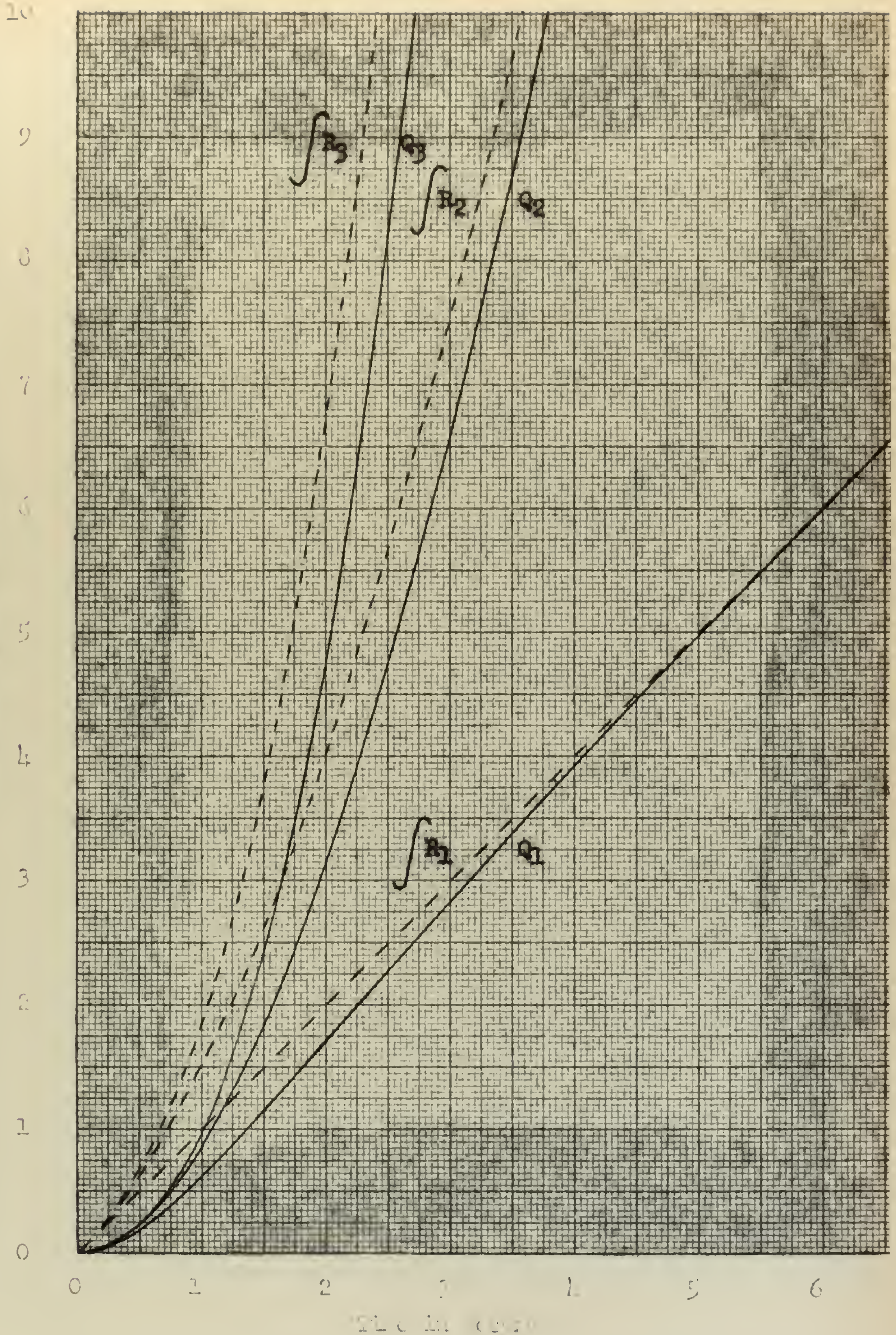


Figure 30. [illegible] fully Weighted [illegible].

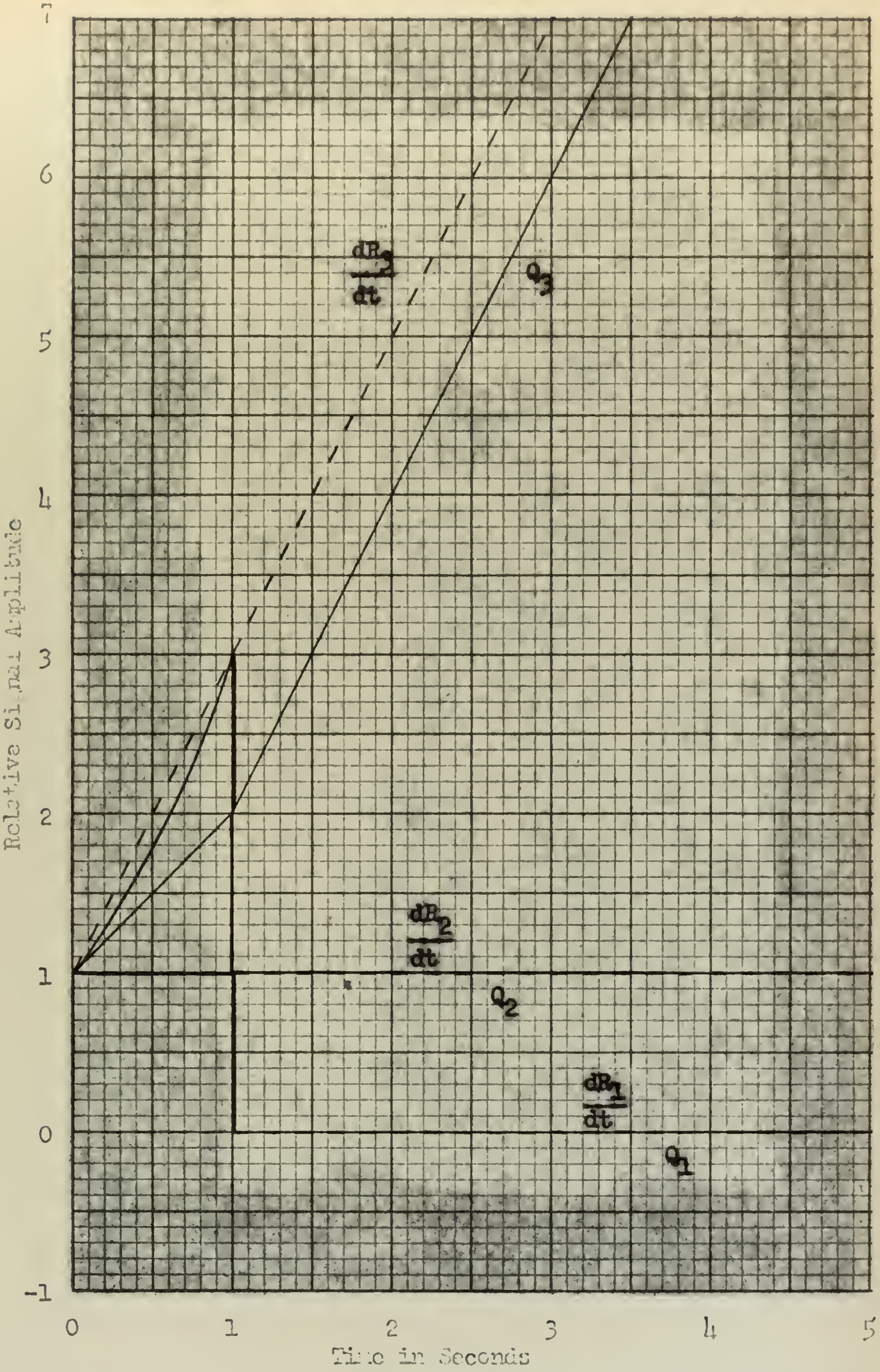


Figure 31. Response of Differentiator Derived from Rectangularly Weighted Smoothing Filter for a Constant.

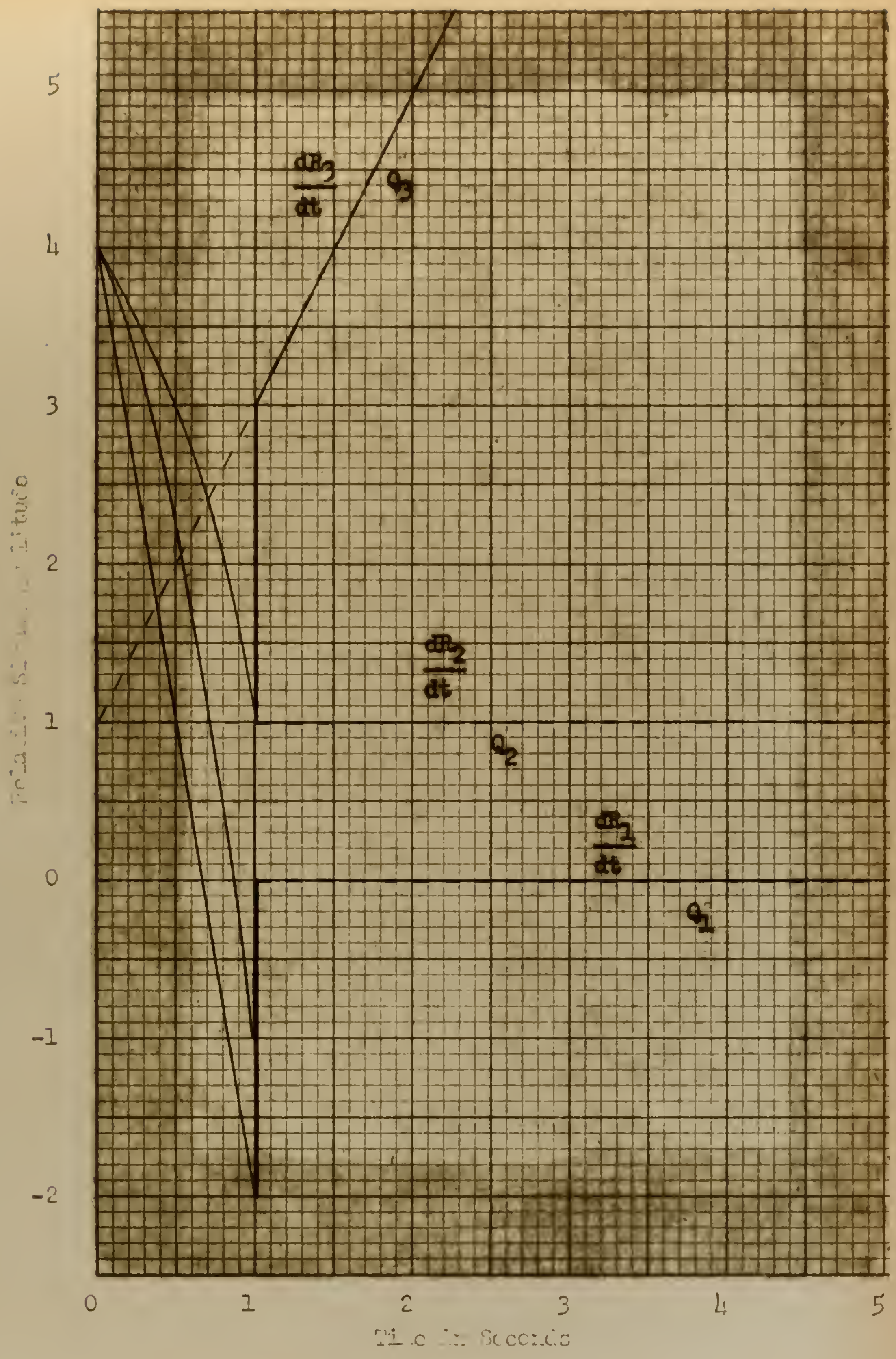


Figure 32. Response of Differentiator Derived from Rectangularly Weighted Smoothing Filter for a Ramp.

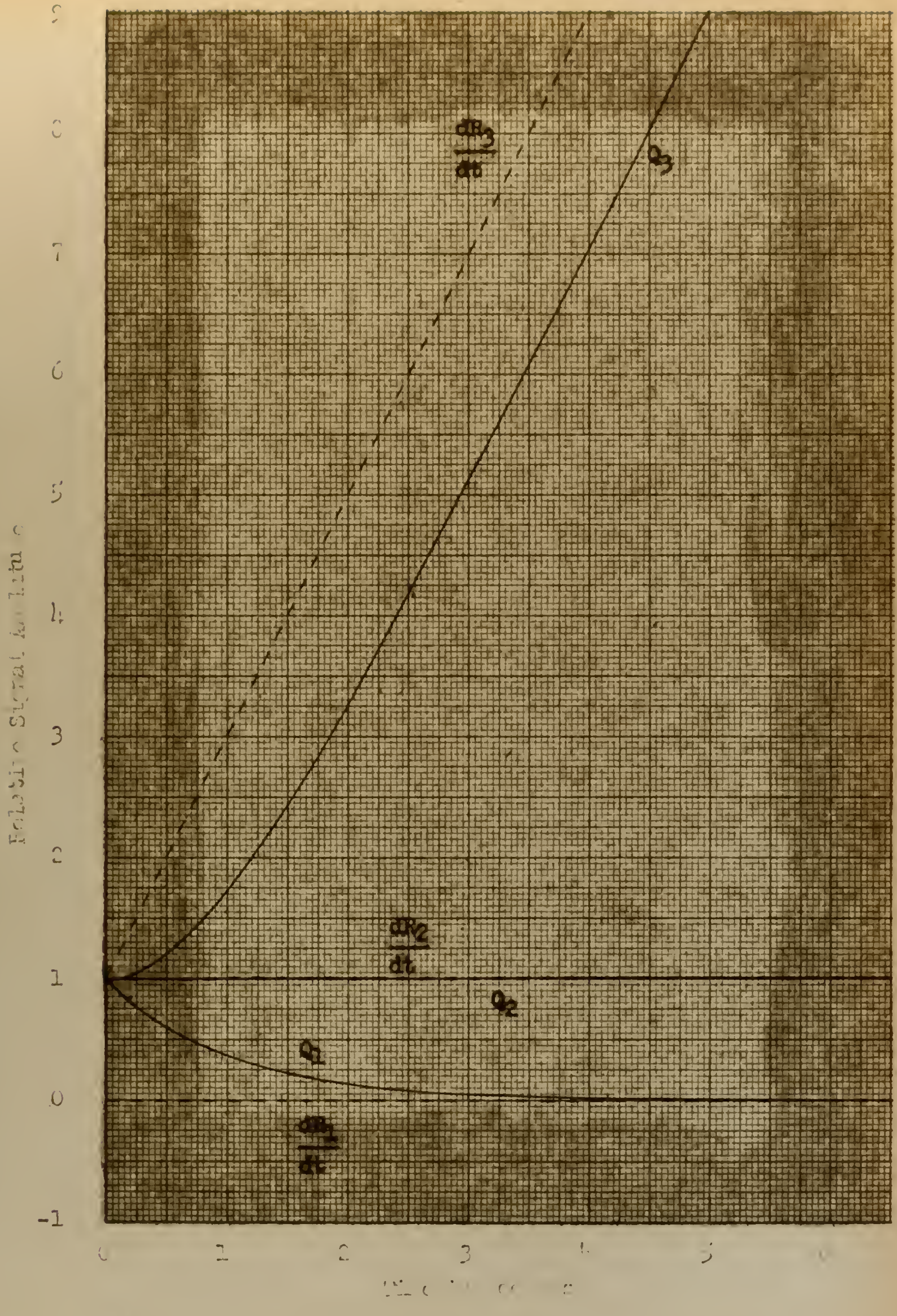

9
8
7
6
5
4
3
2
1
0
-1
0
1
2
3
4
5
6
Relative Signal Amplitude
Time in [illegible]
$\frac{dR_3}{dt}$
Q_3
$\frac{dR_2}{dt}$
Q_2
Q_1
$\frac{dR_1}{dt}$

Figure 23. [illegible]

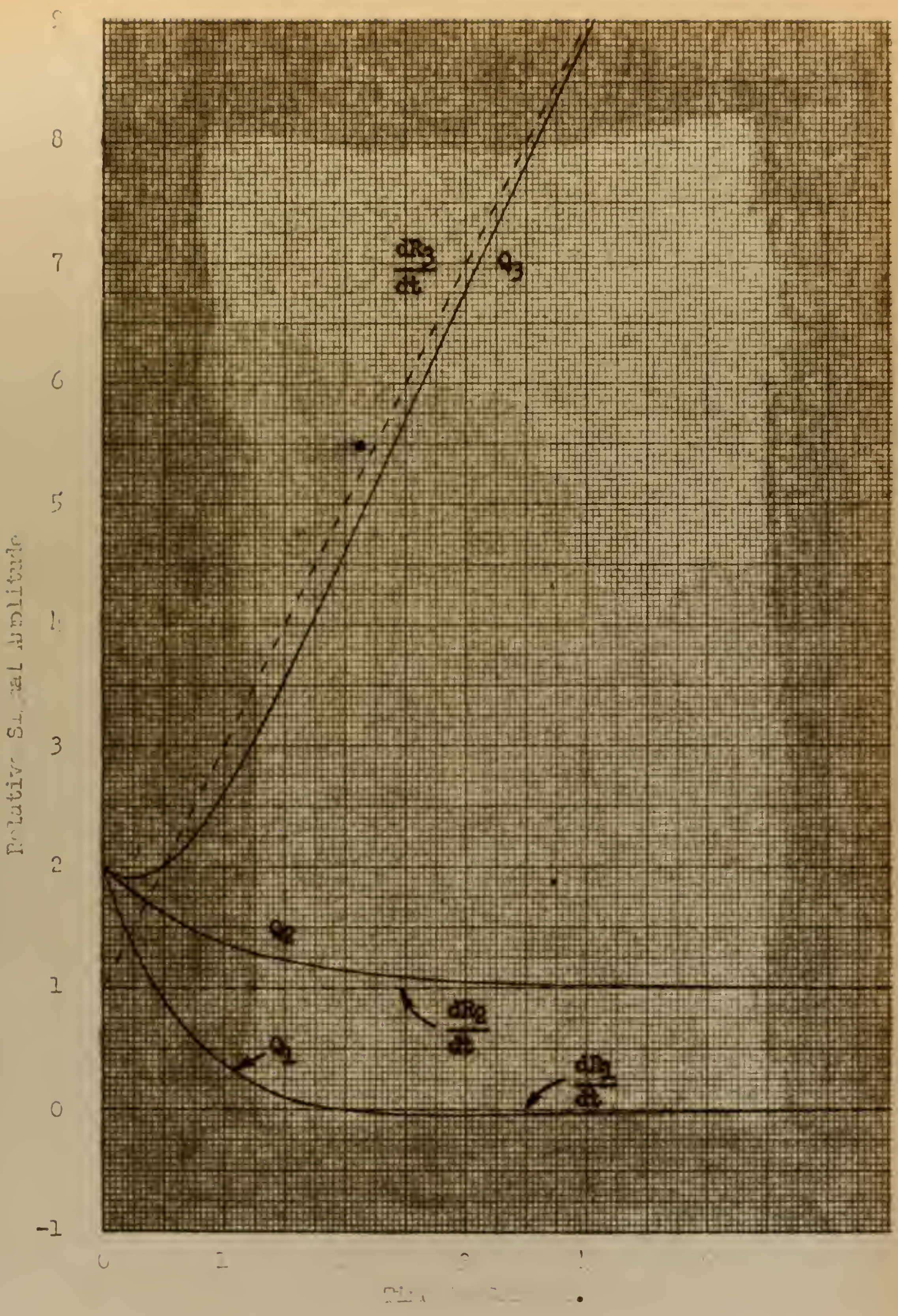

Relative Signal Amplitude
9
8
7
6
5
4
3
2
1
0
-1
0
1
3
4
Time
$\frac{dR_3}{dt}$
Q_3
Q_2
$\frac{dR_2}{dt}$
Q_1
$\frac{dR_1}{dt}$

Figure [illegible]. [illegible]

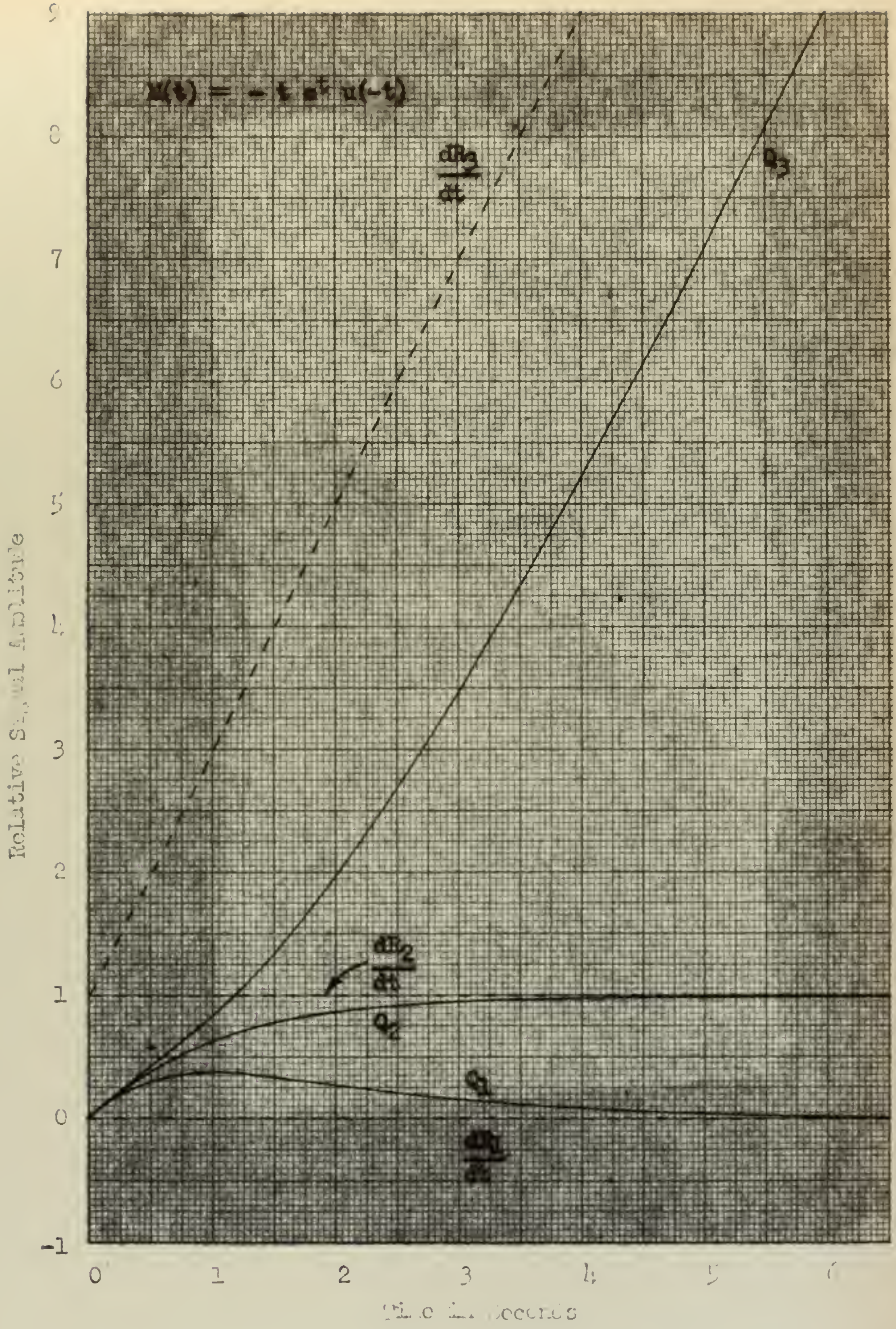


Figure 35. Response of Differentiator Derived from Smoothing Filter Form Constructed With Time [illegible].

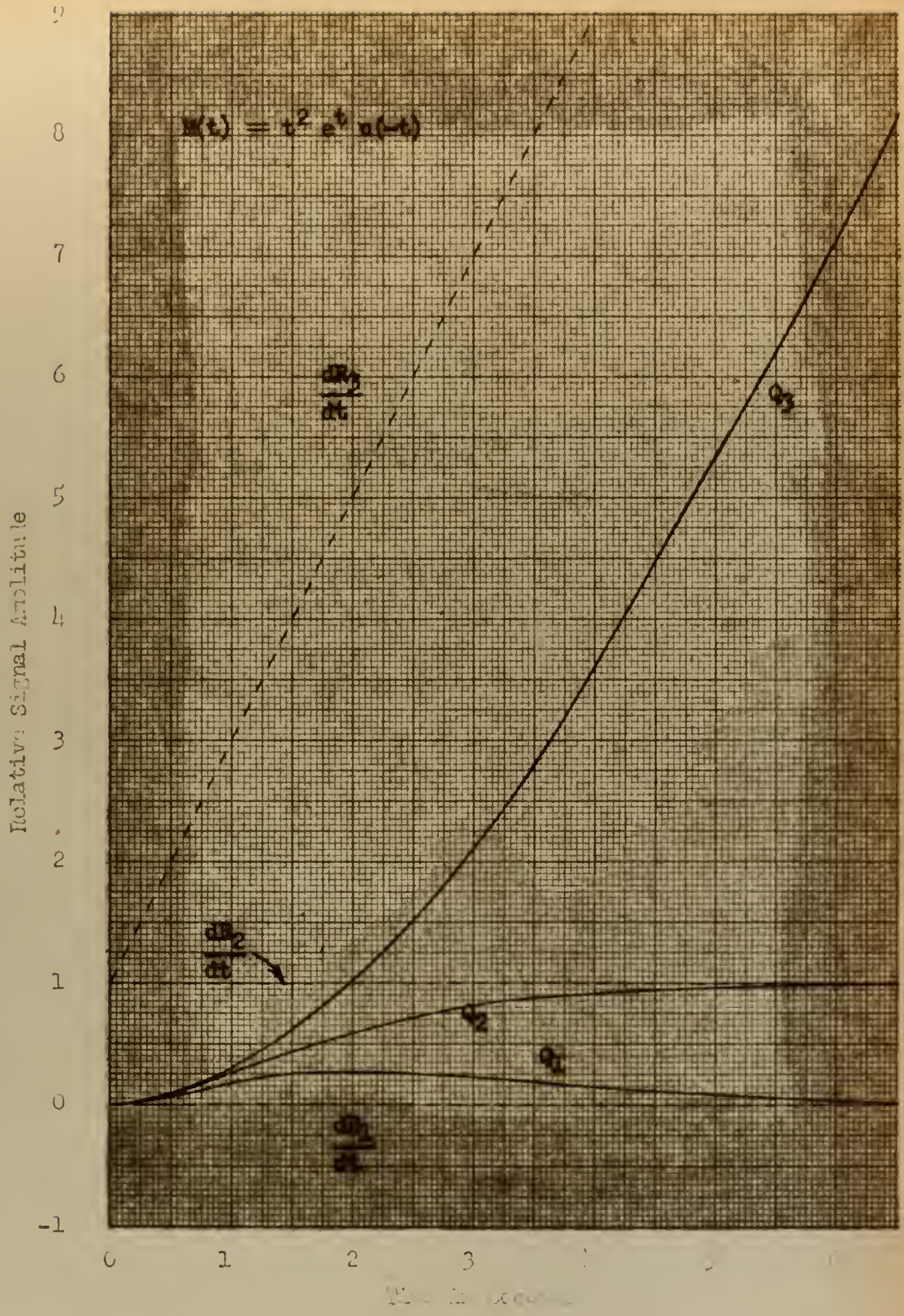

$H(t) = t^2 e^t u(-t)$
$\frac{dR_3}{dt}$
Q_3
$\frac{dR_2}{dt}$
Q_2
Q_1
$\frac{dR_1}{dt}$
Relative Signal Amplitude
9
8
7
6
5
4
3
2
1
0
-1
0
1
2
3
Time in seconds

Figure 36. [illegible]

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